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Parameterized complexity dichotomy for Steiner Multicut ☆

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ABSTRACT

We consider the Steiner Multicut problem, which asks, given an undirected graph G, a collection $\mathcal{T} = \{T_1, \ldots, T_t\}$, $T_i \subseteq V(G)$, of terminal sets of size at most p, and an integer k, whether there is a set S of at most k edges or nodes such that of each set T_i at least one pair of terminals is in different connected components of $G - S$. We provide a dichotomy of the parameterized complexity of Steiner Multicut. For any combination of k, t, p, and the treewidth $\mathrm{tw}(G)$ as constant, parameter, or unbounded, and for all versions of the problem (edge deletion and node deletion with and without deletable terminals), we prove either that the problem is fixed-parameter tractable, W[1]-hard, or (para-)NP-complete. Our characterization includes a dichotomy for Steiner Multicut on trees as well as a polynomial time versus NP-hardness dichotomy (by restricting $k, t, p, \mathrm{tw}(G)$ to constant or unbounded).

1. Introduction

Graph cut problems are among the most fundamental problems in algorithmic research. The classic result in this area is the polynomial-time algorithm for the s–t cut problem of Ford and Fulkerson [32] (independently proven by Elias et al. [28] and Dantzig and Fulkerson [23]). This result inspired a research program to discover the computational complexity of this problem and of more general graph cut problems. One well-studied generalization of the s–t cut problem is the Multicut problem, in which we want to disconnect t pairs of nodes instead of just one pair. In a recent major advance of the research program on graph cut problems, Bousquet et al. [10] and Marx and Razgon [51] showed that Multicut is fixed-parameter tractable in the size k of the cut only, meaning that it has an algorithm running in time $f(k) \cdot \mathrm{poly}(|V(G)|)$ for some

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Table 1
Summary of known and new complexity results for Steiner Multicut, where new results are marked with *; the other entries are either known or follow easily from known results in the literature. Only maximal FPT results and minimal W[·]- or NP-hardness results are listed; empty cells are dominated by other results. E.g. Edge Steiner Multicut with parameter t is hard, since it is already NP-hard for $t = 3$, $p = 2$. For Node Steiner Multicut, one also has to apply the rule that $k < t$ (see Sect. 2) to generate a full characterization of all cases. Tree diagrams of this table are offered in Appendix B.

Constants	Params	Edge Steiner MC	Node Steiner MC	Restr. Node Steiner MC
k	–	poly (Sect. 2)	poly (Sect. 2)	poly (Sect. 2)
$t \le 2$	–	poly (Sect. 2)		poly (Sect. 2)
$t = 3,\ p = 2$	–	NP-h [22]		NP-h [22]
–	k, t	*FPT (Theorem 1.1)	*W[1]-h (Theorem 1.2)	*W[1]-h (Theorem 1.2)
–	k, p, t		FPT (Sect. 2)	FPT (Sect. 2)
t	k			FPT (Sect. 2)
$p = 2$	k	FPT [10,51]	FPT [10,51]	FPT [10,51]
$p = 3$, tw $= 2$	k	*W[1]-h (Theorem 1.3)	*W[1]-h (Theorem 1.3)	*W[1]-h (Theorem 1.3)
–	t, tw	*FPT (Theorem 5.2)	*FPT (Theorem 5.2)	*FPT (Theorem 5.2)
tw $= 1$	–		*poly (Theorem 7.1)	
tw $= 1$	k	*W[2]-h (Theorem 7.2)		*W[2]-h (Theorem 7.2)
tw $= 1$	k, p	*FPT (Theorem 7.4)		*FPT (Theorem 7.4)
tw $= 1$, $p = 2$	–	NP-h [12]		NP-h [12]
tw $= 2$, $p = 2$	–		NP-h [12]	

function f, resolving a longstanding problem in parameterized complexity (with many papers [36,48,50,54] building up to this result).

In this paper, we continue the research program on generalized graph cut problems, and consider the Steiner Multicut problem. This problem was proposed by Klein et al. [43], and appears in several versions, depending on whether we want to delete edges or nodes, and whether we are allowed to delete terminal nodes. Formally, these versions of the Steiner Multicut problem are defined as follows:

{Edge, Node, Restricted Node} Steiner Multicut	
Input:	An undirected graph G with terminal sets $T_1, \ldots, T_t \subseteq V(G)$, and an integer $k \in \mathbb{N}$.
Question:	Find a set S of k {edges, nodes, non-terminal nodes} such that for $i = 1, \ldots, t$ and at least one pair $u, v \in T_i$ there is no $u - v$ path in $G - S$.

Observe that Multicut is the special case of Steiner Multicut in which each terminal set has size two. In general, the terminal sets of Steiner Multicut can have arbitrary size, and we use p to denote $\max_i |T_i|$.

The complexity of Steiner Multicut has been investigated extensively, but so far only from the perspective of approximability. This line of work was initiated by Klein et al. [43], who gave an LP-based $O(\log^3(kp))$-approximation algorithm. The approximability of several variations of the problem has also been considered [3,34,56]; in particular, Garg et al. [34] give an $O(\log t)$-approximation algorithm for Multicut. On the hardness side, even Multicut is APX-hard [12,22] and cannot be approximated within any constant factor assuming the Unique Games Conjecture [14]. We also remark that Steiner cuts (the case when $t = 1$) are of interest: they are an ingredient in several LP-based approximation algorithms (for example for Steiner Forest [1,44]) and there is a connection to the number of edge-disjoint Steiner trees that each connect all terminals [47]. To the best of our knowledge, however, Steiner Multicut in its general form has not yet been considered from the perspective of parameterized complexity.

1.1. Our contribution

In this paper, we fully chart the (parameterized) complexity landscape of Steiner Multicut according to k, t, p (defined as above), and the treewidth tw(G). For all three versions of Steiner Multicut, for each possible combination of k, t, p, and tw(G), where each may be either chosen as a constant, a parameter, or unbounded, we consider the complexity of Steiner Multicut. We show a complete dichotomy: either we provide a fixed-parameter algorithm with respect to the chosen parameters, or we prove a W[1]-hardness or (para-)NP-completeness result that rules out a fixed-parameter algorithm (unless many canonical NP-complete problems have subexponential- or polynomial-time algorithms respectively).

The dichotomy is composed of three main results, along with many smaller ones (see Table 1). These three main results are stated in the three theorems below:

Theorem 1.1. Edge Steiner Multicut *is fixed-parameter tractable for the parameter* $k + t$.

Theorem 1.2. Node Steiner Multicut *and* Restr. Node Steiner Multicut *are* W[1]*-hard for the parameter* $k + t$.