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# Win-win kernelization for degree sequence completion problems ☆

Vincent Froese [1], André Nichterlein *, Rolf Niedermeier

*Institut für Softwaretechnik und Theoretische Informatik, TU Berlin, Germany*

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ABSTRACT

We study provably effective and efficient data reduction for a class of NP-hard graph modification problems based on vertex degree properties. We show fixed-parameter tractability for NP-hard graph completion (that is, edge addition) cases while we show that there is no hope to achieve analogous results for the corresponding vertex or edge deletion versions. Our algorithms are based on transforming graph completion problems into efficiently solvable number problems and exploiting $f$-factor computations for translating the results back into the graph setting. Our core observation is that we encounter a win-win situation: either the number of edge additions is small or the problem is polynomial-time solvable. This approach helps in answering an open question by Mathieson and Szeider [*JCSS* 2012] concerning the polynomial kernelizability of Degree Constraint Edge Addition and leads to a general method of approaching polynomial-time preprocessing for a wider class of degree sequence completion problems.



## 1. Introduction

We propose a general approach for achieving polynomial-size problem kernels for a class of graph completion problems where the goal graph has to fulfill certain degree properties. Thus, we explore and enlarge results on provably effective polynomial-time preprocessing for these NP-hard graph problems. To a large extent, the initial motivation for our work comes from studying the NP-hard graph modification problem Degree Constraint Editing($S$) for non-empty subsets $S \subseteq \{v^-, e^+, e^-\}$ of editing operations ($v^-$: "vertex deletion", $e^+$: "edge addition", $e^-$: "edge deletion") as introduced by Mathieson and Szeider [34].[2] The definition reads as follows.

Degree Constraint Editing($S$) (DCE(S))
**Input:** An undirected graph $G = (V, E)$, two integers $k, r > 0$, and a "degree list function" $\tau : V \to 2^{\{0,\dots,r\}}$.
**Question:** Is it possible to obtain a graph $G' = (V', E')$ from $G$ using at most $k$ editing operations of type(s) as specified by $S$ such that $\deg_{G'}(v) \in \tau(v)$ for all $v \in V'$?

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* Corresponding author.
*E-mail addresses:* vincent.froese@tu-berlin.de (V. Froese), andre.nichterlein@tu-berlin.de (A. Nichterlein), rolf.niedermeier@tu-berlin.de (R. Niedermeier).

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2 Mathieson and Szeider [34] originally introduced a weighted version of the problem, where the vertices and edges can have positive integer weights incurring a cost for each editing operation. Here, we focus on the unweighted version.

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In our work, the set $S$ always consists of a single editing operation. Our studies focus on the two most obvious parameters: the number $k$ of editing operations and the maximum allowed degree $r$. We will show that, although all three variants are NP-hard, DCE($e^+$) is amenable to a generic kernelization method we propose. This method is based on dynamic programming solving a corresponding number problem and $f$-factor computations. For DCE($e^-$) and DCE($v^-$), however, we show that there is little hope to achieve analogous results.

*Previous Work.* There are basically two fundamental starting points for our work. First, there is our previous theoretical work on degree anonymization[3] in social networks [26] motivated and strongly inspired by a preceding heuristic approach due to Liu and Terzi [31] (also see Clarkson et al. [11] for an extended version). Indeed, our previous work for degree anonymization inspired empirical work with encouraging experimental results [25]. A fundamental contribution of this work now is to systematically reveal what the problem-specific parts (tailored towards degree anonymization) and what the "more general" parts of that approach are. In this way, we develop this approach into a general method of wider applicability for a number of graph completion problems based on degree properties. The second fundamental starting point is Mathieson and Szeider's work [34] on (weighted) DCE($S$). They showed several exponential-size problem kernels for the operations vertex deletion and edge deletion. For the case that the degree list of each vertex contains only one number, they even obtain a polynomial-size problem kernel. They left open, however, whether it is possible to reduce DCE($e^+$) in polynomial time to a problem kernel of size polynomial in $r$—we will affirmatively answer this question. Indeed, while Mathieson and Szeider provided several fixed-parameter tractability results with respect to the combined parameter $k$ and $r$, they partially left open whether similar results can be achieved using the stronger parameterization[4] with the single parameter $r$. Recently, Golovach [20] described kernelization and fixed-parameter results for closely related graph editing problems where vertex and edge deletions and edge insertions are allowed, the degree list of each vertex contains exactly one number, and the resulting graph has to be connected.

From a more general perspective, all these considerations fall into the category of "graph editing to fulfill degree constraints", which recently received significant interest in terms of parameterized complexity analysis [1,17,20,35].

*Our Contributions.* Answering an open question of Mathieson and Szeider [34], we present an $O(kr^2)$-vertex kernel for DCE($e^+$) which we then transfer into an $O(r^5)$-vertex kernel using a strategy rooted in previous work [26,31]. A further main contribution of our work in the spirit of meta kernelization [2] is to clearly separate problem-specific from problem-independent aspects of this strategy, thus making it accessible to a wider class of degree sequence completion problems. We observe that if the goal graph shall have "small" maximum degree $r$, then the actual graph structure is in a sense negligible and thus allows for a lot of freedom that can be algorithmically exploited. This paves the way to a *win-win situation* of either having guaranteed a small number of edge additions or the overall problem being solvable in polynomial-time anyway—another example in the list of win-win situations exploited in parameterized algorithmics [16].

Besides our positive kernelization results, we exclude polynomial-size problem kernels for DCE($e^-$) and DCE($v^-$) subject to the assumption that NP $\nsubseteq$ coNP/poly, thereby showing that the exponential-size kernel results by Mathieson and Szeider [34] are essentially tight. In other words, this demonstrates that in our context edge completion is much more amenable to kernelization than edge deletion or vertex deletion are. We also prove NP-hardness of DCE($v^-$) and DCE($e^+$) for graphs of maximum degree three, implying that the maximum degree is not a useful parameter for parameterized complexity or kernelization purposes. Last but not least, we develop a general preprocessing approach for Degree Sequence Completion problems which yields a search space size that is polynomially bounded in the parameter. While this per se does not give polynomial kernels, we derive fixed-parameter tractability with respect to the combined parameter maximum degree and solution size. The usefulness of our method is illustrated by further example degree sequence completion problems.

*Notation.* All graphs in this paper are undirected, loopless, and simple (that is, without multiple edges). For a graph $G = (V, E)$, we set $n := |V|$ and $m := |E|$. The degree of a vertex $v \in V$ is denoted by $\deg_G(v)$, the *maximum vertex degree* by $\Delta_G$, and the *minimum vertex degree* by $\delta_G$. For a finite set $U$, we denote by $\binom{U}{2}$ the set of all size-two subsets of $U$. We denote by $\overline{G} := (V, \binom{V}{2} \setminus E)$ the *complement graph* of $G$. For a vertex subset $V' \subseteq V$, the subgraph induced by $V'$ is denoted by $G[V']$. For an edge subset $E' \subseteq \binom{V}{2}$, $V(E')$ denotes the set of all endpoints of edges in $E'$ and $G[E'] := (V(E'), E')$. For a set $E'$ of edges with endpoints in a graph $G$, we denote by $G + E' := (V, E \cup E')$ the graph that results from inserting all edges in $E'$ into $G$. Similarly, we define for a vertex set $V' \subseteq V$, the graph $G - V' := G[V \setminus V']$. For each vertex $v \in V$, we denote by $N_G(v) := \{u \in V \mid \{u, v\} \in E\}$ the *open neighborhood* of $v$ in $G$ and by $N_G[v] := N_G(v) \cup \{v\}$ the *closed neighborhood*. We omit subscripts if the corresponding graph is clear from the context. A vertex $v \in V$ with $\deg(v) \in \tau(v)$ is called *satisfied* (otherwise *unsatisfied*). We denote by $\mathcal{U} \subseteq V$ the set of all unsatisfied vertices, formally $\mathcal{U} := \{v \in V \mid \deg_G(v) \notin \tau(v)\}$.

[3] For a given integer $k$, the task here is to add as few edges as possible to a graph such that the resulting graph is *k-anonymous*, that is, there is no vertex degree in the graph which occurs at least one but less than $k$ times.

[4] Fixed-parameter tractability with respect to the parameter $r$ (trivially) implies fixed-parameter tractability with respect to the combined parameter $(k, r)$, but the reverse clearly does not hold in general; see Komusiewicz and Niedermeier [29] for a broader discussion in this direction.