Journal of Computer and System Sciences 82 (2016) 213–228

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

Editing to Eulerian graphs ☆

Konrad K. Dabrowski [a,*], Petr A. Golovach [b], Pim van 't Hof [b], Daniël Paulusma [a]

[a] School of Engineering and Computing Sciences, Durham University, Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
[b] Department of Informatics, University of Bergen, PB 7803, 5020 Bergen, Norway

ARTICLE INFO

Article history:
Received 25 October 2014
Received in revised form 11 September 2015
Accepted 9 October 2015
Available online 11 November 2015

Keywords:
Eulerian graphs
Graph editing
Polynomial algorithm

ABSTRACT

The Eulerian Editing problem asks, given a graph G and an integer k, whether G can be modified into an Eulerian graph using at most k edge additions and edge deletions. We show that this problem is polynomial-time solvable for both undirected and directed graphs. We generalize these results for problems with degree parity constraints and degree balance constraints, respectively. We also consider the variants where vertex deletions are permitted. Combined with known results, this leads to full complexity classifications for both undirected and directed graphs and for every subset of the three graph operations.

1. Introduction

Graph modification problems play a central role in algorithmic graph theory, partly due to the fact that they naturally arise in numerous practical applications. A graph modification problem takes as input a graph G and an integer k, and asks whether G can be modified into a graph belonging to a prescribed graph class $\mathcal{H}$, using at most k operations of certain allowed types. The most common operations that are considered in this context are edge additions ($\mathcal{H}$-Completion), edge deletions ($\mathcal{H}$-Edge Deletion), vertex deletions ($\mathcal{H}$-Vertex Deletion), and a combination of edge additions and edge deletions ($\mathcal{H}$-Editing). The intensive study of graph modification problems has produced a plethora of classical and parameterized complexity results (see e.g. [2–8,10,13,15–17,19,21–23,25,26]).

An undirected (resp. directed) graph G is Eulerian if it contains a walk that begins and ends with the same vertex and that uses every edge (resp. arc) exactly once. As an immediate consequence, an undirected graph is Eulerian if and only if it is connected and every vertex has even degree. Similarly, a directed graph is Eulerian if it is strongly connected[1] and balanced, i.e. the in-degree of every vertex equals its out-degree. Eulerian graphs form a well-known graph class both within algorithmic and structural graph theory.

Several groups of authors have investigated the problem of deciding whether or not a given undirected graph can be made Eulerian using a small number of operations. Boesch et al. [2] presented a polynomial-time algorithm for Eulerian Completion, and Cai and Yang [5] showed that the problems Eulerian Vertex Deletion and Eulerian Edge Deletion are NP-complete [5]. When parameterized by the number k of allowed operations, it is known that Eulerian Vertex Deletion

☆ The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013)/ERC Grant Agreement no. 267959 and from EPSRC Grant EP/K025090/1. An extended abstract of this paper appeared in the proceedings of FSTTCS 2014 [9].
* Corresponding author.
E-mail addresses: konrad.dabrowski@durham.ac.uk (K.K. Dabrowski), petr.golovach@ii.uib.no (P.A. Golovach), pim.vanthof@ii.uib.no (P. van 't Hof), daniel.paulusma@durham.ac.uk (D. Paulusma).

[1] Replacing "strongly connected" by "weakly connected" yields an equivalent definition of Eulerian digraphs, as it is well-known that a balanced digraph is strongly connected if and only it is weakly connected (see e.g. [8]).

http://dx.doi.org/10.1016/j.jcss.2015.10.003

is W[1]-hard [5], while Eulerian Edge Deletion is fixed-parameter tractable [8]. Cygan et al. [8] showed that the classical and parameterized complexity results for Eulerian Vertex Deletion and Eulerian Edge Deletion also hold for the directed variants of these problems. Recently, Goyal et al. [17] improved the fixed-parameter tractability results of Cygan et al. [8] for the directed and undirected variants of Eulerian Edge Deletion by giving algorithms with running times that are single-exponential in k. The same authors also proved that the Undirected Connected Odd Edge Deletion problem, which asks whether it is possible to obtain a connected graph in which all vertices have odd degree by deleting at most k edges, is fixed-parameter tractable when parameterized by k.

Another problem that can be seen as involving editing to an Eulerian multigraph is the Chinese Postman problem, also known as the Route Inspection problem [20]. In this problem a connected graph G, together with an integer k, is given and the question is whether or not there exists a closed walk in G that uses every edge of G at least once and that has length at most $|E(G)|+k$. In other words, can we add a total of at most k copies of existing edges to G in order to modify G into an Eulerian multigraph? Edmonds and Johnson [12] showed that both the undirected and directed variant of this problem can be solved in polynomial time. The Rural Postman problem generalizes the Chinese Postman problem, as it requires that only every edge of some subset of $E(G)$ needs to be used at least once in the closed walk. Dorn et al. [10] proved that the Rural Postman is fixed-parameter tractable for directed multigraphs when parameterized by the number of arcs that may be added.

Our contribution We generalize, extend and complement known results on graph modification problems dealing with Eulerian graphs and digraphs. The main contribution of this paper consists of two non-trivial polynomial-time algorithms: one for solving the Eulerian Editing problem, and one for solving the directed variant of this problem. Given the aforementioned NP-completeness result for Eulerian Edge Deletion and the fact that $\mathcal{H}$-Editing is NP-complete for many graph classes $\mathcal{H}$ [3,26], we find it particularly interesting that Eulerian Editing turns out to be polynomial-time solvable. To the best of our knowledge, the only other natural non-trivial graph class $\mathcal{H}$ for which $\mathcal{H}$-Editing is known to be polynomial-time solvable is the class of split graphs [18].

In fact, our polynomial-time algorithms are implications of two more general results. In order to formally state these results, we need to introduce some terminology. Let ea, ed and vd denote the operations edge addition, edge deletion and vertex deletion, respectively. For any set $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$ and non-negative integer k, we say that a graph G can be (S,k)-*modified* into a graph H if H can be obtained from G by using at most k operations from S. We define the following problem for every $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$:

CDPE(S):	Connected Degree Parity Editing(S)
Instance:	A (simple) graph G, an integer k and a function $\delta: V(G) \to \{0,1\}$.
Question:	Can G be (S,k)-modified into a connected graph H with $d_H(v) \equiv \delta(v) \pmod 2$ for each $v \in V(H)$?

Inspired by the work of Cygan et al. [8] on directed Eulerian graphs, we also study a natural directed variant of the CDBE(S) problem. Denoting the in- and out-degree of a vertex v in a digraph G by $d_G^{in}(v)$ and $d_G^{out}(v)$, respectively, we define the following problem for every $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$:

CDBE(S):	Connected Degree Balance Editing(S)
Instance:	A (simple) digraph G, an integer k and a function $\delta: V(G) \to \mathbb{Z}$.
Question:	Can G be (S,k)-modified into a weakly connected digraph H with $d_H^{out}(v) - d_H^{in}(v) = \delta(v)$ for each $v \in V(H)$?

In Section 3, we prove that CDPE(S) can be solved in polynomial time when $S=\{\mathsf{ea}\}$ and when $S=\{\mathsf{ea},\mathsf{ed}\}$. The first of these two results extends the result by Boesch et al. [2] on Eulerian Completion and the second yields the first polynomial-time algorithm for Eulerian Editing, as these problems are equivalent to CDPE($\{\mathsf{ea}\}$) and CDPE($\{\mathsf{ea},\mathsf{ed}\}$), respectively, when we set $\delta \equiv 0$ (i.e. when $\delta(v)=0$ for every $v \in V(G)$). The complexity of the problem drastically changes when vertex deletion is allowed: we prove that for every subset $S \subseteq \{\mathsf{ea},\mathsf{ed},\mathsf{vd}\}$ with $\mathsf{vd} \in S$, the CDPE(S) problem is NP-complete and W[1]-hard with parameter k, even when $\delta \equiv 0$. This complements results by Cai and Yang [5] stating that CDPE(S) is NP-complete and W[1]-hard with parameter k when $S=\{\mathsf{vd}\}$ and $\delta \equiv 0$ or $\delta \equiv 1$. Our results, together with the