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Complete binary trees embeddings in Möbius cubes

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ABSTRACT

The complete binary tree as an important network structure has long been investigated for parallel and distributed computing, which has many nice properties and used to be embedded into other interconnection architectures. The Möbius cube M_n is an important variant of the hypercube Q_n. It has many better properties than Q_n with the same number of edges and vertices. In this paper, we prove that the complete binary tree with $2^n - 1$ vertices can be embedded with dilation 1, congestion 1, load 1 into M_n and expansion tending to 1.

1. Introduction

Interconnection architectures play a key role in parallel computing systems. An interconnection network can be represented by a simple graph $G = (V, E)$, where V is the vertex set and E is the edge set. In this paper, we also use $V(G)$ and $E(G)$ to denote V and E, respectively.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, an embedding from G_1 to G_2 is a mapping $\psi : V_1 \rightarrow V_2$. We call G_1 the *guest graph* and G_2 the *host graph* with respect to the embedding ψ. Many applications, such as architecture simulation, processor allocation, VLSI chip design can be modeled as a *graph embedding* problem [1–5].

There are four important metrics to measure the performance of an embedding, namely, *dilation*, *expansion*, *load* and *congestion* [6]. The dilation of embedding ψ is defined as $dil(G_1, G_2, \psi) = \max\{dist(G_2, \psi(u), \psi(v)) \mid (u, v) \in E_1\}$, which measures the communication delay, where $dist(G_2, \psi(u), \psi(v))$ denotes the distance between the two vertices $\psi(u)$ and $\psi(v)$ in G_2. The expansion of embedding ψ is defined as $exp(G_1, G_2, \psi) = |V(G_2)|/|V(G_1)|$, which measures the processor utilization. The congestion of embedding ψ is defined as $C(G_1, G_2, \psi) = \max\{C(e) \mid e \in E_2\}$, which measures queuing delay of messages, where $C(e)$ denotes the number of edges in G_1 mapped to a path in G_2 that includes e. To measure the processing time of tasks is referred to as the load in an embedding. The load of embedding ψ is defined as $load(G_1, G_2, \psi) = \max\{load(v) \mid v = \psi(u), u \in V_1\}$, where $load(v)$ denotes the number of vertices of G_1 mapped on v.

The hypercubes are the most popular interconnection architectures, because they have many advantageous properties such as comparably lower vertex degree and diameter, higher connectivity and symmetry. Vaidya et al. introduced a class of hypercube-like networks [7], which consists most of hypercube variants, such as crossed cubes [8], Möbius cubes [9],

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twisted cubes [10], and parity cubes [11], etc. The n-dimensional Möbius cube (denoted by M_n) was proposed first by Cull and Larson. It has many attractive features superior to those of n-dimensional hypercubes (denoted by Q_n). For example, M_n has a diameter with about half that of Q_n. Xu and Deng [12] proved that the wide diameter of M_n is about half of Q_n's. Fan [13] showed that any cycle of length l ($4 \leq l \leq 2^n$) can be embedded into M_n with dilation 1 ($n \geq 2$), which is not possessed by Q_n.

Paths, cycles, meshes and trees are the common networks often used as guest graphs in many graph embeddings [14–21,34,35]. So far, embeddings of cycles, paths and starlike trees into hypercube-like networks have been studied in [15,22–24]. According to the results in them, one can derive the corresponding embeddings of cycles, paths and starlike trees into Möbius cubes. Recently, an algorithm to construct independent spanning trees on Möbius cubes has been proposed [35], which actually shows that there exist n independent spanning trees rooted at an arbitrary vertex can be embedded with dilation 1 into M_n.

Due to the desirable performances and wide applications of the complete binary tree, its embeddability into various interconnection architectures is significant. So far, much work about the embeddings of the complete binary tree into meshes, star graphs, lines, grids, and butterfly network has appeared in the literature [25–29]. While there have been only a few studies that have attempted to embed the complete binary tree into hypercube variants. [30] presented dilation 1 embedding of complete binary trees in recursive circulant $G(2^m, 4)$. In [31], [32] and [37] it was proved that the complete binary tree can be embedded with dilation 1 into folded cubes, enhanced cubes, crossed cubes, and parity cubes, respectively, while it was proved that the complete binary tree can be embedded into a hypercube with either expansion 2 and dilation 1 or with expansion 1 and dilation 2 [33].

In this paper, we study the embedding of complete binary trees into Möbius cubes. Firstly, we prove that the complete binary trees can be embedded into M_n for any odd integer $n \geq 5$. Furthermore, we discuss the embedding into M_n for any even integer $n \geq 6$. Finally, it is proved that the $2^n - 1$ vertices complete binary tree can be embedded with dilation 1, congestion 1, load 1 into M_n and expansion tending to 1.

The rest of this paper is organized as follows: Section 2 provides the preliminaries. Section 3 proves that the complete binary tree with $2^n - 1$ vertices can be embedded with dilation 1 into the n-dimensional Möbius cube and expansion tending to 1. The final section concludes this paper.

2. Preliminaries

In this section, we will give some terminologies, definitions and basic lemmas used in this paper. Given two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, G_2 is said to be a *subgraph* of G_1 if $V_2 \subseteq V_1$ and $E_2 \subseteq E_1$. The subgraph *induced* by V' in G_1 is denoted by $G_1[V']$, where $V' \subseteq V_1$. For two subgraphs G' and G'' of G_1 with $V(G') \cap V(G'') = \emptyset$, $G_1[V(G') \cup V(G'')]$ can be denoted by $G' \cup G''$. An *isomorphism* from G_1 to G_2 is a bijection $\phi : V(G_1) \to V(G_2)$ such that $(u, v) \in E_1$ if and only if $(\phi(u), \phi(v)) \in E_2$ for any two vertices $u, v \in V_1$. If G_1 is *isomorphic* to G_2, we write as $G_1 \cong G_2$.

For any integer $n \geq 1$, a binary string x of length n is denoted by $x_n x_{n-1} \ldots x_i \ldots x_2 x_1 (1 \leq i \leq n)$, where $x_i \in \{0, 1\}$ is said to be the ith bit of x and $x_n \ldots x_k (2 \leq k \leq n)$ is called a prefix of x, and furthermore, x can be written as $(x_n \ldots x_k) x_{k-1} \ldots x_2 x_1$. The ith bit of x can also be denoted as $bit(x, i)$. The *complement* of x_i will be denoted by $\overline{x_i} = 1 - x_i$. The length of x will be denoted by $|x|$. Given two binary strings x and y of the same length, if the string value of x is less than that of y in the alphabet order, then we write $x < y$. For example, $0011 < 0100$. Let y be a binary string with $|y| = m$. y^j denotes a new binary string by repeating j times to the string y, where $0 \leq j \leq m$, and particularly, y^0 denotes the empty string. We use e_i to denote an n-bit binary string whose i-th bit is 1 and all the other bits are 0, where $1 \leq i \leq n$.

The n-dimensional Möbius cube has the same vertex number and edge number as the n-dimensional hypercube. Every vertex of M_n is identified by a unique binary string of length n. M_n has two types, one 0-type n-dimensional Möbius cube (denoted by 0-M_n) and another 1-type n-dimensional Möbius cube (denoted by 1-M_n). Either 0-M_n or 1-M_n may be denoted by M_n. We adopt the recursive definition of the Möbius cube from [36].

Definition 1. (See [36].) 0-M_1 and 1-M_1 are both the complete graph on two vertices whose labels are 0 and 1. For any integer $n \geq 2$, both 0-M_n and 1-M_n contain one 0-type $(n-1)$-dimensional sub-Möbius cube M_n^0 and one 1-type $(n-1)$-dimensional sub-Möbius cube M_n^1. The first bit of every vertex of M_n^0 is 0; the first bit of every vertex of M_n^1 is 1. For two vertices $x = x_n x_{n-1} \ldots x_1 \in V(M_n^0)$ and $y = y_n y_{n-1} \ldots y_1 \in V(M_n^1)$, where $x_n = \overline{y_n} = 0$,

(1) $(x, y) \in E(0\text{-}M_n)$ if and only if $x_i = y_i$, $i = 1, 2, \ldots, n-1$;
(2) $(x, y) \in E(1\text{-}M_n)$ if and only if $x_i = \overline{y_i}$, $i = 1, 2, \ldots, n-1$.

Fig. 1(a) and (b) illustrate 0-M_4 and 1-M_4, respectively. By Definition 1, we can easily check whether a given pair of vertices are adjacent in M_n.

Lemma 2. *For any integer $n \geq 1$, $(u_n u_{n-1} \ldots u_2 u_1, v_n v_{n-1} \ldots v_2 v_1)$ is an edge of M_n if and only if there exists an integer i with $1 \leq i \leq n$ such that*

(1) *$v_j = u_j$ for any integer j with $i + 1 \leq j \leq n$,*
(2) *$v_i = \overline{u_i}$,*