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Bases of closure systems over residuated lattices ☆

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ABSTRACT

We present results on bases of closure systems over residuated lattices, which appear in applications of fuzzy logic. Unlike the Boolean case, the situation is not straightforward as there are two non-commuting generating operations involved. We present a decomposition theorem for a general closure operator and utilize it for computing generators and bases of the closure system. We show that bases are not unique and may in general have different sizes, and obtain a constructive description of the size of a largest base. We prove that if the underlying residuated lattice is a chain, all bases have the same size.

1. Introduction

Closure systems play a crucial role in various parts of mathematics and computer science, including algebra, logic, programming, databases, data analysis and management of data in general. It is well known that every closure system $\mathcal{S}$ in a set U, i.e. a system $\mathcal{S} \subseteq 2^U$ closed with respect to arbitrary intersections, has a unique base. This base, i.e. an inclusion-minimal subsystem of $\mathcal{S}$ that generates $\mathcal{S}$, consists of all the $\bigcap$-irreducible sets in $\mathcal{S}$—sets that cannot be obtained as intersections of other sets in $\mathcal{S}$. More generally, one may ask for a base of the closure system $[\mathcal{T}]$ generated by a given $\mathcal{T} \subseteq 2^U$. Again, such a base is unique and consists of the $\bigcap$-irreducible members of $\mathcal{T}$. Bases of closure systems, or dually, interior systems, are frequently encountered. In formal concept analysis, for instance, determining for a given input binary relation I between objects and attributes a minimal relation J such that the concept lattice of J, i.e. the lattice of all fixpoints of the Galois connection induced by J, is isomorphic to the concept lattice of I, is easily rephrased as the problem of determining bases of closure systems [6,8]. Another example, dual in that closure systems are replaced by interior systems, is the row/column-base of a Boolean matrix, which is just the base of the interior system generated by sets whose characteristic vectors are just the matrix rows/columns [14].

In this paper, we study bases of **L**-closure systems [2], which naturally appear when instead of bivalent (0–1, yes-or-no) data, the problems at hand involve data with grades from a partially ordered scale L (see e.g. [3–5,9,11,13]). In such a setting, sets or—more precisely—characteristic functions $A : U \to \{0, 1\}$ of sets are replaced by their generalizations $A : U \to L$, which are called L-sets, and the two-element Boolean algebra on $\{0, 1\}$ underlying the set calculus is replaced by its appropriate generalization **L**. In accordance with modern fuzzy logic [7,11–13], we take for **L** an arbitrary residuated lattice, leaving Boolean algebras, Heyting algebras, MV-algebras, BL-algebras, and other structures as special cases. In particular, if $L = \{0, 1\}$ then **L** becomes the two-element Boolean algebra and **L**-closure systems may be identified with ordinary closure systems. General **L**-closure systems represent a non-trivial generalization of ordinary ones. In addition to the fact that

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L involves intermediary values, i.e. those between 0 and 1, a substantial difference consists in the fact that while the only generating operation in ordinary closure systems is intersection, in **L**-closure systems there are two such operations: (infimum-based) intersection of L-sets and (residuum-based) multiplication of L-sets by constants in L, which is degenerate in the bivalent case $L = \{0, 1\}$. The properties of these operations, in particular the fact that the induced operators do not commute (multiplications of intersections is not the same as intersections of multiplications), make the problem of bases of **L**-closure systems non-trivial. For example, as we show in this paper, bases are not unique and in general, may have different sizes.

2. Preliminaries: residuated lattices, L-sets, L-closure systems

Recall that a (*complete*) *residuated lattice* [3,7,13,16] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the lattice order is denoted by $\leq$; 0 and 1 denote the least and greatest element, respectively);
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Throughout the paper, **L** denotes an arbitrary complete residuated lattice. Common examples of complete residuated lattices include those defined on the real unit interval or on a finite chain $L = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$. For instance, for $L = [0, 1]$, we can use any left-continuous t-norm for $\otimes$, such as minimum, product, or Łukasewicz, and the corresponding residuum $\rightarrow$. Residuated lattices are commonly used in fuzzy logic [3,11,13], where the grades $a \in L$ are interpreted as degrees of truth and the operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of the (truth function of) conjunction and implication, respectively. The only residuated lattice with $L = \{0, 1\}$ coincides with the two-element Boolean algebra of classical logic in which case $\otimes$ and $\rightarrow$ are the truth functions of classical conjunction and implication. For more information on residuated lattices we refer to [3,7,13,16].

Given **L**, one may consider L-sets, i.e. generalizations of (characteristic functions of) ordinary sets, and their algebra. An L-*set* (or L-fuzzy set) in a universe set U is a mapping $A : U \rightarrow L$ assigning to every $u \in U$ an element $A(u) \in L$ interpreted as the truth degree to which u belongs to A. The set of all L-sets in U, denoted by L^U, is equipped with operations extending the classical set operations. For instance, the $\wedge$-based intersection is defined by $(A \wedge B)(u) = A(u) \wedge B(u)$. For $A, B \in L^U$, the degree of inclusion of A in B is defined by $S(A, B) = \bigwedge_{u\in U}(A(u) \rightarrow B(u))$. If $S(A, B) = 1$, which is equivalent to $A(u) \leq B(u)$ for each $u \in U$, one writes $A \subseteq B$ and says that A is included in B. For more details on L-sets, we refer to [3,11,13].

A system $\mathcal{S} \subseteq L^U$ is called an **L**-*closure system* in U [2] if

– $\mathcal{S}$ is closed under *left* $\rightarrow$*-multiplications*, i.e. $a \rightarrow A \in \mathcal{S}$ for each $a \in L$ and $A \in \mathcal{S}$;
– $\mathcal{S}$ is closed under $\bigwedge$*-intersections*, i.e. if $A_j \in \mathcal{S}$ $(j \in J)$ then $\bigwedge_{j\in J} A_j \in \mathcal{S}$.

Here, $a \rightarrow A$ and $\bigwedge_{j\in J} A_j$ are defined by

$$(a \rightarrow A)(u) = a \rightarrow A(u) \quad \text{and} \quad (\textstyle\bigwedge_{j\in J} A_j)(u) = \bigwedge_{j\in J} A_j(u)$$

for any $u \in U$.

Remark 1. Since $0 \rightarrow a = 1$ and $1 \rightarrow a = a$ for any $a \in L$, we have $0 \rightarrow A = U$ and $1 \rightarrow A = A$ for each $A \in L^U$. Moreover, since U is the intersection of the empty system of L-sets, it belongs to any **L**-closure system. Therefore, if **L** is the two-element Boolean algebra, **L**-closure systems are just systems closed under $\bigwedge$-intersections and may be identified with ordinary closure systems. As a result, the notion of an **L**-closure system generalizes the ordinary notion of a closure system.

Let us also note that, analogously to the ordinary case, **L**-closure systems in U are in one-to-one correspondence with **L**-closure operators in U, i.e. mappings $C : L^U \rightarrow L^U$ satisfying

$$A \subseteq C(A), \quad S(A, B) \leq S(C(A), C(B)), \quad \text{and} \quad C(A) = C(C(A)),$$

for every $A, B \in L^U$. Namely, **L**-closure systems are just the sets of fixed points of **L**-closure operators [2].

3. Results

3.1. Bases of L-closure systems

Obviously, the set L^U is an **L**-closure system in U (the largest one) and one can easily see that an intersection of an arbitrary system of **L**-closure systems is an **L**-closure system. It hence follows from classical results [6] that for every system