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Interval-valued fuzzy coimplications and related dual interval-valued conjugate functions

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ABSTRACT

The aim of this paper is to introduce the dual notion of interval conjugate implications, the interval coimplications, as interval representations of corresponding conjugate fuzzy coimplications. Using the canonical representation, this paper considers both the correctness and the optimality criteria, in order to provide interpretation for fuzzy coimplications as the non-truth degree of conditional rule in expert systems and study the action of interval automorphisms on such interval fuzzy connectives. It is proved that interval automorphisms acting on $\mathbb{N}$-dual interval coimplications preserve the main properties of interval implications discussed in the literature including the duality principle. Lastly, the action of interval automorphisms on interval classes of border, model and S-coimplications are considered, summarized in commutative diagrams.

1. Introduction

Fuzzy logic provides the theoretical foundation for reasoning about imprecise propositions, such reasoning has been referred to as approximate reasoning, using the fundamental idea of fuzzy set membership function to deal with partial truths. The membership function of ordinary fuzzy sets are often precise, requiring each element of the universal set be assigned to a real number. However, in some concepts and context the value of membership degree might include uncertainty considering a fuzzy connective degree not by an exact number but rather by an interval of possible values, or, even more generally, by a fuzzy value from the interval [0; 1]. Thus, it may be able to define membership function only approximately, identifying meaningful lower and upper bounds of membership grades of an element in the universal set. In such approach, a membership function is assigned to a closed interval of real numbers between the identified lower and upper bounds. Fuzzy sets identified by such membership functions are called interval-valued fuzzy sets.

Interval-valued fuzzy logic was first proposed by Sambuc [1] under the name ϕ-fuzzy sets. Interval-valued fuzzy sets appear in the literature in many ways, see, e.g., [2–8].

Interval-valued fuzzy sets are a particular case of type-2 fuzzy sets, which have been studied by Zadeh [9] and others authors (e.g., [10,11]) since the 70's, allowing to deal not only with vagueness (lack of sharp class boundaries), but also with uncertainty (lack of information) [12,13]. Since then, the integration of fuzzy theory with interval mathematics considers different viewpoints, as properly pointed out by Lodwick [13] (see also in [12–23]) generating several different approaches.

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1.1. Interval-valued non-truth degree in conditional rules

In order to provide interpretation of the interval non-truth degree of conditional rules, consider $\mathbb{U} = \{[a,b] \mid 0 \leqslant a \leqslant b \leqslant 1\}$. Let X be an ordinary set. An interval fuzzy set $\mathbb{A}$ on X is given by $\mathbb{A} = \{\langle x, \mu_{\mathbb{A}}(x)\rangle \mid x \in X\}$, where $\mu_{\mathbb{A}} : X \to \mathbb{U}$.

When $\mathbb{A}$ and $\mathbb{B}$ are interval fuzzy sets on X and Y, respectively, the intervals $\mu_{\mathbb{A}}(x)$ and $\mu_{\mathbb{B}}(y)$ denote, respectively, the interval membership degree of the element x in $\mathbb{A}$ and of the element y in $\mathbb{B}$.

Let x and y be variables taking values in X and Y, respectively. A conditional rule in expert systems has the following form

$$\text{if } x \text{ is } \mathbb{A} \quad \text{then } y \text{ is } \mathbb{B} \tag{1}$$

and it is interpreted as an interval fuzzy relation using an interval implication function $\mathbb{I} : X \times Y \to \mathbb{U}$, associating to each $(x, y) \in X \times Y$, $\mathbb{I}(\mu_{\mathbb{A}}(x), \mu_{\mathbb{B}}(y))$, which is always interpreted as the interval truth degree of the conditional rule (1).

The definition of interval-valued fuzzy coimplication has been considered in [24,25] to analyze the non-truth degree of such conditional rule. Thus, in a fuzzy dual-based approach, when $\mathbb{J} : X \times Y \to \mathbb{U}$ is an interval coimplication function, $\mathbb{J}(\mu_{\mathbb{A}}(x), \mu_{\mathbb{B}}(y))$ provides an interpretation to the interval non-truth degree of such antecedent-consequent form of a conditional rule, see [26] and [27].

While a fuzzy implication is an extension of the Boolean implication ($p \Rightarrow q$), meaning that p is sufficient for q, a coimplication is an extension of the Boolean coimplication ($p \nLeftarrow q$), meaning that p is not necessary for q.

1.2. Related works

Although the fuzzy coimplications can be used in a dual approach of fuzzy implications, they have been less discussed in the literature. However, the coimplications also play an important role in classical logic, fuzzy logic and intuitionistic fuzzy logic. See also [25,28–31].

In [32] and [33], the concept of fuzzy coimplication was introduced as a new approach to approximate reasoning of expert systems using the equivalence relation for modus ponens of the inference in fuzzy expert systems instead of fuzzy implication.

The algebraic properties of fuzzy S-implications and coimplications and also residual implications and coimplications are studied in [26], in order to provide a theoretical background for approximate reasoning applications.

Based on [24] and [34], a class of intuitionistic fuzzy implication operators can be constructed by using fuzzy implications (I), fuzzy negations (N), N-dual fuzzy implications (I_N) and aggregation operators. In such case, an intuitionistic fuzzy conditional rule always has associated to it a pair (I, I_N), such that $I + I_N \leqslant 1$.

Thus, it seems natural to extend this notion of fuzzy coimplication to interval-valued fuzzy theory.

1.3. Main contribution of this work

This paper follows the approach first introduced in Bedregal and Takahashi's works [35,36], applied in other papers, where interval extensions for some fuzzy connectives were provided (see, e.g., [37–41]), considering both correctness (accuracy) and optimality aspects [42].

Extending the constructions introduced in [43], where the interval coimplications are considered, this paper introduces the interval-valued extension related to the conjugate of an interval fuzzy coimplications, based on the canonical representation of a real function proposed in [35] and [42]. We focus on the discussion of properties preserved by the action of interval automorphisms on interval-valued fuzzy implications and coimplications, including their connections with the duality principle. The canonical interval representation related to the classes of border and model implications are also considered.

In addition, we study the class of interval S-coimplications, which can be obtained as $\mathbb{N}$-dual construction of interval S-implications in the class of interval border implications, taking $\mathbb{N}$ as a strong interval fuzzy negation and $\mathbb{S}_{\mathbb{N}}$ as the interval $\mathbb{N}$-dual structure of an interval t-conorm $\mathbb{S}$.

1.4. Outline of this paper

This paper is organized as follows. Firstly, Section 2 compresses preliminaries: Subsection 2.1 presents the main concepts of interval representations of real functions; Subsection 2.2 studies the canonical representation of automorphisms and the interval conjugate of an interval function; Subsection 2.3 considers main concepts of fuzzy negations and duality relationship; and Subsection 2.4 reports the main definitions related to interval triangular norms and conorms. In sequence, the main definitions of interval fuzzy coimplications, their N-dual structures and some properties usually demanded from a fuzzy coimplication J (implication I) are naturally extended to an interval-based approach in Section 3.

The interdependencies among these properties, the suitability of interval $\mathbb{N}$-dual fuzzy connectives on $\mathbb{U}$ and the duality between interval-valued implications and coimplications, which is preserved by canonical representation, including the main related results are also verified. Classes of interval border, model and S-coimplications are introduced in Subsection 3.3.