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Structural operational semantics for continuous state stochastic transition systems

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ABSTRACT

In this paper we show how to model syntax and semantics of stochastic processes with continuous states, respectively as algebras and coalgebras of suitable endofunctors over the category of measurable spaces **Meas**. Moreover, we present an SOS-like rule format, called *MGSOS*, representing abstract GSOS over **Meas**, and yielding *fully abstract universal semantics*, for which behavioral equivalence is a congruence. An MGSOS specification defines how semantics of processes are composed by means of *measure terms*, which are expressions specifically designed for describing finite measures. The syntax of these measure terms, and their interpretation as measures, are part of the MGSOS specification. We give two example applications, with a simple and neat MGSOS specification: a "quantitative CCS", and a calculus of processes living in the plane $\mathbb{R}^2$ whose communication rate depends on their distance. The approach we follow in these cases can be readily adapted to deal with other quantitative aspects.

1. Introduction

Process algebras are widely used for compositional modeling of nondeterministic, communicating, mobile systems. States of these systems are represented by syntactic-defined terms $P, Q, \ldots$; the semantics is represented by means of a (labelled) transition relation of the form $P \xrightarrow{\alpha} Q$ between these terms, where labels represent what can be observed by the environment. According to the Structural Operational Semantics (SOS) paradigm [38], this relation is specified by a set of inference rules whose application is driven by the syntactic structure of processes. In order to guarantee important properties about the resulting semantics, several formats for these SOS specifications have been studied. A well-known format is the so-called GSOS [13], which guarantees the bisimilarity to be a congruence. This framework is particularly appealing because it makes languages easier to understand, compare, and extend. In particular, a process algebra can be easily extended with new operators, without the need of time-consuming and error-prone proofs of congruence results.

In recent years this successful approach has been applied also to *stochastic* and *probabilistic* systems, which have received increasing attention due to their important applications to performance evaluation, systems biology, etc. [27,14,26,21]. In order to deal with these quantitative aspects, the transition relation has to be modified by adding some real-valued parameter; often it has the form $P \xrightarrow{\alpha,r} Q$, meaning that "$P$ can do α and continue as Q, with probability (or rate) r". Bartels [10] and Klin and Sassone [31] have investigated rule formats for discrete probabilistic and stochastic systems, respectively,

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which guarantee probabilistic/stochastic bisimilarity to be a congruence. This approach has been further generalized in [34], with a rule format for general discrete, non-deterministic *weighted* labelled transition systems.

However, these formats do not cover the case of systems whose behavior is influenced by some continuous data, which may change as the system evolves. Typical examples are systems with spatial/geometric informations (e.g., in wireless networks, distance may affect data access and rates; in biological models, diffusion alters the signaling pathways, etc.), or intensional parameters like temperature, pressure, concentrations, etc.. A proper representation of the states of these systems must include these continuous data, hence the state space is a *continuous* set, and a term language representing these states cannot be countable any longer; see e.g. [15,30]. Formally, this means that we cannot use transitions of the form $P \xrightarrow{\alpha,r} Q$, because the rate of reaching a precise Q from P may be zero, yet the rate of reaching any neighborhood of Q may be nonzero.

This leads us to consider transitions of the form $P \xrightarrow{\alpha} \mu$, where μ is a (finite) *measure* of the possible outcomes of P, as in [12]. This measure specifies a quantitative (probabilistic, stochastic, ...) information about the different transitions. Stochastic calculi with a similar transition format have been considered also in [16,6] for dealing with specific equational stochastic systems, and with behavioral equivalences which are proved to be congruences. However, differently from the case of discrete processes, these SOS specifications and results are rather *ad hoc*, not based on any general framework for operational descriptions.

In order to cover this gap, in [7] we introduced a new rule format for *probabilistic* systems with continuous states, which guarantees that the resulting probabilistic behavioral equivalence is a congruence. In this paper we continue this line of research, by focusing on *stochastic* systems. More precisely, we present a GSOS rule format for stochastic systems with continuous states, which ensures that the resulting stochastic behavioral equivalence is a congruence.

In order to achieve this goal, we follow the *bialgebraic* approach [44]. This approach relies on the fact that the syntax of processes can be represented as the initial algebra of a suitable signature functor, and their semantics as coalgebras of a suitable "behavioral" functor. The key idea of the bialgebraic framework is that rule specification systems can be formulated in terms of certain natural transformations called *distributive laws* between the signature and the behavioral functors. These distributive laws allow us to define a *fully abstract* denotational semantics with respect to the behavioral equivalence: two processes have the same denotation if and only if their behavior is the same. Moreover, this equivalence is a congruence and (if the behavioral functor preserves weak pullbacks) coincides with bisimilarity.

Usually, the bialgebraic construction has been applied in the **Set** or variations thereof like presheaf categories [44,23,24,33]. This construction has been applied also to *discrete* probabilistic and stochastic systems: in [10], labelled probabilistic transition systems are shown to be coalgebras of the functor $(\mathcal{D}_\omega(\_)+1)^L : \mathbf{Set} \to \mathbf{Set}$; in [31] labelled stochastic transition systems are shown to be coalgebras of the functor $\mathcal{R}_\omega(\_)^L : \mathbf{Set} \to \mathbf{Set}$ where L is the set of labels, and $\mathcal{R}_\omega$ is the functor over **Set** such that $\mathcal{R}_\omega(X)$ is the set of finitely supported measures over X. A more uniform and general treatment in the case of *weighted* labelled transition systems has been developed in [34].

However, in order to apply this approach to our setting we have to solve several issues. The crucial point is that our semantics have to deal with the probability of *sets* of possible outcomes, rather than of single states. This corresponds to move to the category **Meas** of *measurable spaces* and *measurable functions*, as advocated in [20,22,37]; for instance, the behavior of a stochastic system with continuous state can be modeled as a coalgebra of the functor $\Delta(\_)^L : \mathbf{Meas} \to \mathbf{Meas}$, where Δ associates with any measurable space (X, Σ) the set of finite measures over it.

Porting the bialgebraic approach from **Set** to **Meas** is not straightforward. First, the behavioral functor Δ does not preserve weak pullbacks [36,48]; in fact, bisimilarity is strictly included in behavioral equivalence [43]. Hence, we focus on behavioral equivalence instead of bisimilarity.

Secondly, **Meas** is not known to be Cartesian closed; as a consequence, most constructions which can be carried out on **Set** and other topoi cannot be ported easily to **Meas**. In particular, we cannot follow Bartels's approach for deriving a rule format from a distributive law [10], because the extra structure given by σ-algebras does not allow one to decompose natural transformations of complex type as collections of natural transformations of simpler type.

Moreover, a SOS rule format should be as easy to apply, and as syntactic, as possible. In particular, it has to define a system's behavior in terms of those of its subsystems. In traditional GSOS format, this is reflected by the fact that the target of a transition is a process built from the components of the source process, and their corresponding semantics. In our settings, the target of a transition $P \xrightarrow{\alpha} \mu$ is not a process term, but a real-valued function over some measurable space, without any syntactic structure to leverage. In order to circumvent this problem, we propose to use *measure terms*, i.e., syntactic expressions purposely introduced to denote measures. The syntax of measure terms, and their interpretation as measures, is part of the specification. This new degree of freedom allows us to capture many examples from the literature.

Summarizing, we present a new operational semantics specification format, which we call *Measure GSOS format*; a MGSOS specification is given by a set of rules for deriving transitions of the form $P \xrightarrow{\alpha} \mu$ where both P and μ are syntactic objects (of possibly different languages), together with an interpretation of these measure terms into measures. We will show that any LTS specification in this format leads to a distributive law of type $S(Id \times \Delta^L) \Rightarrow (\Delta T^S)^L$, where S is the syntactic functor and T^S the corresponding free monad. As a consequence, the induced behavioral equivalence is always a congruence.

This paper is the extended and revised version of the conference paper [7]. Some original contributions of this version are: the generalization of the format to the stochastic case, several proofs and new examples and discussions (see Section 7); moreover, in Section 2.3 we provide new tools to prove the existence of initial algebras and final coalgebras in categories different from **Set**, but with some well-behaved factorization system. We apply these tools in Section 3.3 to prove that the