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Relational presheaves, change of base and weak simulation

Paweł Sobociński

ECS, University of Southampton, UK

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ABSTRACT

We show that considering labelled transition systems as relational presheaves captures several recently studied examples in a general setting. This approach takes into account possible algebraic structure on labels. We show that left (2-)adjoints to change-of-base functors between categories of relational presheaves with relational morphisms always exist and, as an application, that weak closure (in the sense of Milner) of a labelled transition system can be understood as a left adjoint to a change-of-base functor.

1. Introduction

A famous application of coalgebra [3,4] is as an abstract setting for the study of labelled transition systems (LTS). Indeed, an LTS with label set A is a coalgebra for the functor $\mathcal{P}(A \times -)$. LTSs are thus the objects of the category of coalgebras for this functor. The arrows, in concurrency theoretical terminology, are functional bisimulations. This category of coalgebras (modulo size issues) has a final object that gives a canonical notion of equivalence, although other approaches are available in general.[1] Ordinary bisimulations can be understood as spans of coalgebra morphisms. The coalgebraic approach has been fruitful: amongst many notable works we mention Turi and Plotkin's elegant approach to structural operational semantics congruence formats via bialgebras [50].

In another influential approach, Winskel and Nielsen [52] advocated the use of presheaf categories as a general semantic universe for the study of labelled transition systems. Morphisms turn out to be functional simulations, functional *bi*simulations can be characterised as open maps with respect to a canonical (via the Yoneda embedding) choice of path category [24]. Ordinary bisimulations are then spans of open maps, with some side conditions.

Both the coalgebraic approach and the presheaf approach have generated much subsequent research and have found several applications that we do not account for here. Concentrating on the theory of labelled transition systems, there are some limitations to both approaches. For example both take for granted that the set of labels A is monolithic and has no further structure. In fact, several labelled transition systems have "sets" of labels that are monoids [9] or even categories [30,16,35]. Such examples are more challenging to capture satisfactorily with the aforementioned approaches but some progress has been made—for instance, Bonchi and Montanari [7] captured labelled transition systems on reactive systems (in the sense of Leifer and Milner [30]) as certain coalgebras on presheaves.

There is also a mismatch between notions typically studied by concurrency theorists or researchers in the operational semantics of concurrent languages and the morphisms in categories of coalgebras or in presheaves. From the point of

E-mail address: ps@ecs.soton.ac.uk.

[1] See [48] for an overview of the notions that appear in the literature.

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view of process theory, the morphisms in the aforementioned categories are not the notions typically studied: *functional* simulations and *functional* bisimulations[2] instead of ordinary simulations and ordinary bisimulations.

Finally, and perhaps most importantly, the most natural notions of equivalence in applications are often weak (in the sense of Milner) and these tend to be technically challenging to capture in the coalgebra or presheaf settings. One can think of weak bisimulation as ordinary bisimulation on a labelled transition system that has been "saturated" with the silent τ-actions, but this is just another way of saying that τ is made the identity of a monoid of actions. We will develop this idea in Section 6. Because the coalgebraic and presheaf approaches were not designed with a view to accommodate such *algebraic structure* on the set of labels, some work has to be performed in order to talk about weak equivalences, see for example [47,15].

This paper proposes the universe of relational presheaves[3] as an abstract setting for the study of labelled transition systems. Relational presheaves are lax functors [28] (or, equivalently, morphisms of bicategories [5]) $\mathbf{C}^{op} \to \mathbf{Rel}$, where **Rel** is the 2-category of relations with objects sets, arrows relations, and 2-cells inclusions of relations (in particular, the hom-categories are partial orders—some authors refer to such 2-categories as locally partially ordered). The applications that we study in this paper are related to the previous use of relational presheaves and enriched category theory in automata theory [41,25,26]. Relational presheaves have been called by various other names: specification structures [1], relational variable sets [17], dynamic sets [49], with several applications in Computer Science and related fields. Functors to **Rel** as a way of giving semantics to flowcharts were considered as early as 1972 by Burstall [11].

The typical examples of **C** that we shall consider will be monoids (i.e. categories with one object) and categories of contexts, in the sense of Leifer and Milner [30,45]. Ordinary A-labelled transition systems can be seen as instances of the former by considering the free monoid A^*. A more interesting example where the label set is a non-free monoid is the LTS considered in [9] or the LTSs that arise from expressions in the algebra of Span(**Graph**) [27]. Weak derived LTS for reactive systems in the sense of Jensen [23], are examples of relational presheaves for **C** a category of contexts. Tile logics [16] can also be seen as relational presheaves where **C** is not merely a monoid: here the vertical composition of tiles gives the algebraic structure on the tile transitions.

There are two different, natural choices for morphisms between relational presheaves. One, suggested by the connection with enriched category theory, is to take functional oplax natural transformations, which Rosenthal refers to as rp-morphisms [41, Definition 3.3.2]. Indeed, the category of relational presheaves and rp-morphisms $\mathscr{R}(\mathbf{C})$ is equivalent to the category of categories enriched over the free quantaloid on **C** [41, Theorem 3.3.1]. In our examples rp-morphisms turn out to be functional simulations. Here, change-of-base functors u^* always have left-adjoints and they have right-adjoints whenever u satisfies the weak factorisation lifting property [36]. One consequence is that $\mathscr{R}(\mathbf{C})$ has limits that are computed pointwise.

A second choice for morphisms is to also consider all (i.e. not necessarily functional) oplax natural transformations, which Rosenthal refers to as generalized rp-morphisms [41, Definition 3.3.3]: in our examples these turn out to be *ordinary* simulations. They have been studied to a lesser extent and are the morphisms on which we shall focus in this paper. Following Rosenthal, we denote the 2-category of relational presheaves and generalized rp-morphisms by $\mathscr{R}^*(\mathbf{C})$. Given relational presheaves $h, h' \in \mathscr{R}^*(\mathbf{C})$, morphisms $h \to h'$ organise themselves in a sup-lattice, with the order inherited from **Rel**, and joins are preserved by composition, in other words $\mathscr{R}^*(\mathbf{C})$ is a quantaloid [41, Proposition 3.3.2]. The mathematical universe of relational presheaves with relational morphisms is rich. Most importantly, some familiar constructions from the concurrency theory literature can be characterised as adjunctions. For example, we shall show that the weak-closure of a labelled transition system is actually a left (2-)adjoint to a change-of-base functor. In this sense, this paper continues the programme of [51] in that category theory is used to clarify and distill constructions commonly used in the study of models of concurrency.

Explicitly, the contributions of this paper are:

(i) the insight that the theory of relational presheaves and (generalized) rp-morphisms is a suitable mathematical universe for the study of labelled transition systems whose sets of labels bear algebraic structure.
(ii) the proof that the construction of left adjoints to change-of-base functors in $\mathscr{R}(\mathbf{C})$ [36] works also in the "larger" 2-category of relational presheaves and *generalized* rp-morphisms $\mathscr{R}^*(\mathbf{C})$ (Theorem 5.1). This fact does not seem to have been noticed before.
(iii) as an application of this result, we show how weak-closure of an LTS can be considered as a left 2-adjoint to a suitable change-of-base functor. This construction can be viewed as an instance of a general mathematical theory of weak simulations.

A previous conference version of this article appeared in [46]. In this extended version we add Theorem 5.1, together with a more thorough exposition of examples and the constructions that appeared in the original conference version.

[2] Although Hughes and Jacobs [21] and Hasuo [18,19] have also studied simulations coalgebraically—the latter using oplax morphisms on coalgebras in Kleisli categories; these are conceptually closely related to morphisms between relational presheaves. Recently, Levy [31] has considered similarity coalgebraically using the notion of *relators*, which also have some technical similarities with relational presheaves.

[3] Using Rosenthal's terminology [41].