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Two Cartesian closed categories of information algebras

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ABSTRACT

The compact information algebra and the continuous information algebra are two special information algebras, which are algebraic structures modeling computation in many different contexts. We show that the set of all continuous mappings between two continuous information algebras also forms a continuous information algebra. Further, we obtain that the two categories **COMP** and **CON**, consisting of compact information algebras and continuous information algebras as objects respectively, continuous mappings as morphisms, are both Cartesian closed.

1. Introduction

The information algebra framework was inspired by the formulation of simple axioms that enable local computation [5,10,11]. The compact information algebra given in the study of representation of information is a system such that each piece of information has the property of finite representability. Later the continuous information algebra as a generalization of compact information algebras was introduced. The relationship between continuous information algebras and compact information algebras and properties of ordering structures on these two kinds of information algebras were investigated [3,4].

Continuous mappings between information algebras was presented in [8], which has shown that an information algebra consisting of all continuous mappings between compact information algebras with two appropriate operations combination and focusing is also compact. Naturally, with continuous mappings as morphisms between objects, categories of information algebras are obtained. It is well known that category always is a field related to theoretical study on computer and system sciences [6,15–18]. Particularly, a Cartesian closed category can provide a suitable mathematic model for simply-typed lambda calculus in theoretical computer science [7,12,13]. Therefore, it is worth exploring to find out Cartesian closed categories of information algebras, which could promote closer ties between theoretical computer science and information algebra theory. In this paper we present an equivalent characterization of continuous information algebras, and show that the set of all continuous mappings between continuous information algebras can also form a continuous information algebra. Then both the categories of compact information algebras and continuous information algebras are shown to be Cartesian closed further.

2. Preliminaries

First we recall some basic definitions in information algebra theory. In fact, the framework of information algebras gives a basic mathematical model for describing the modes of information processing. Here we adopt the definition of information algebras presented in [8].

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Definition 2.1. A tuple (Φ, D) is a system with two operations defined, where D is a lattice:

1. Combination: $\Phi \times \Phi \to \Phi$; $(\phi, \psi) \mapsto \phi \otimes \psi$,
2. Focusing: $\Phi \times D \to \Phi$; $(\phi, x) \mapsto \phi^{\Rightarrow x}$.

We impose the following axioms on (Φ, D), and call it an information algebra:

(1) Semigroup: Φ is associative and commutative under combination. There is an element $e \in \Phi$ such that for all $\phi \in \Phi$ with $e \otimes \phi = \phi \otimes e = \phi$. We call e the neutral element.
(2) Transitivity: For $\phi \in \Phi$ and $x, y \in D$, $(\phi^{\Rightarrow y})^{\Rightarrow x} = \phi^{\Rightarrow x \wedge y}$.
(3) Combination: For $\phi, \psi \in \Phi$ and $x \in D$, $(\phi^{\Rightarrow x} \otimes \psi)^{\Rightarrow x} = \phi^{\Rightarrow x} \otimes \psi^{\Rightarrow x}$.
(4) Neutrality: For $x \in D$, $e^{\Rightarrow x} = e$.
(5) Support: For $\phi \in \Phi$, there is an x such that $\phi^{\Rightarrow x} = \phi$.
(6) Idempotency: For $\phi \in \Phi$ and $x \in D, \phi \otimes \phi^{\Rightarrow x} = \phi$.

If (Φ, D) is an information algebra, we write $\psi \leq \phi$ if $\psi \otimes \phi = \phi$. The following lemma gives some basic properties of this order relation on information algebras.

Lemma 2.1. *(See [8].) If (Φ, D) is an information algebra, then for $\phi, \psi \in \Phi$ and $x, y \in D$,*

(1) $\phi^{\Rightarrow x} \leq \phi$;
(2) $\phi \otimes \psi = \sup\{\phi, \psi\}$;
(3) $\phi \leq \psi$ *implies* $\phi^{\Rightarrow x} \leq \psi^{\Rightarrow x}$;
(4) $x \leq y$ *implies* $\phi^{\Rightarrow x} \leq \phi^{\Rightarrow y}$;
(5) $\phi_1 \leq \phi_2$ *and* $\psi_1 \leq \psi_2$ *imply* $\phi_1 \otimes \psi_1 \leq \phi_2 \otimes \psi_2$.

In lattice theory, a non-empty subset A of a partially ordered set L is said to be directed, if for all $a, b \in A$, there is a $c \in A$ such that $a, b \leq c$. For $a, b \in L$, we call a way-below b (see [2]), in symbols $a \ll b$, if and only if for all directed subsets $X \subseteq L$, if $\vee X$ exists and $b \leq \vee X$, then there exists an $x \in X$ such that $a \leq x$. In particular, we call $a \in L$ a finite element if $a \ll a$.

In the following we introduce a kind of information algebras with continuous ordering structures. The examples of continuous information algebras can be referred to [3,4].

Definition 2.2. (See [3,4,8].) Let (Φ, D) be an information algebra, $\Gamma \subseteq \Phi$ is closed under combination and contains the empty information e. Then (Φ, D) is called a continuous information algebra with a basis Γ, if it satisfies the following axioms of convergence and density:

1. Convergency: If $X \subseteq \Gamma$ is directed, then the supremum $\vee X$ exists.
2. Density: $\phi^{\Rightarrow x} = \vee\{\psi \in \Gamma : \psi = \psi^{\Rightarrow x} \ll \phi\}$ for all $\phi \in \Phi$ and $x \in D$.

Moreover, if a continuous information algebra (Φ, Γ, D) satisfies the axiom of compactness, then we call (Φ, Γ, D) a compact information algebra.

3. Compactness: If $X \subseteq \Gamma$ is a directed set, and $\phi \in \Gamma$ such that $\phi \leq \vee X$ then there exists a $\psi \in X$ such that $\phi \leq \psi$.

If (Φ, Γ, D) is a continuous information algebra, by definition, we can easily obtain that the set $\{\psi \in \Gamma : \psi = \psi^{\Rightarrow x} \ll \phi\}$ is directed for all $\phi \in \Phi$ and $x \in D$ by the closedness of Γ under combination.

Lemma 2.2. *(See [8].) Let (Φ, Γ, D) be a compact information algebra, then the following holds:*

(1) $\psi \in \Gamma$ *if and only if* $\psi \ll \psi$.
(2) *If* $\psi \in \Gamma$, *then* $\psi \ll \phi$ *if and only if* $\psi \leq \phi$.

Suppose that L is a complete lattice. If for all $a \in L$, $a = \vee\{x \in L : x \ll a\}$, we call L a continuous lattice. If for all $a \in L$, $a = \vee\{x \in L : x \ll x \leq a\}$, we call L an algebraic lattice (see [2]). For the continuity and the compactness of information algebras, we have the following equivalent forms.

Theorem 2.1. *(See [4].) Let (Φ, D) be an information algebra.*

(1) *(Φ, D) is continuous if and only if $(\Phi, \leq)$ is a complete lattice and for all $\phi \in \Phi$ and $x \in D$,*

$$\phi^{\Rightarrow x} = \vee\left\{\psi \in \Phi : \psi = \psi^{\Rightarrow x} \ll \phi\right\}.$$