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# Products of matrices and recursively enumerable sets

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ABSTRACT

We study connections between products of matrices and recursively enumerable sets. We show that for any positive integers $m$ and $n$ there exist three matrices $M, N, B$ and a positive integer $q$ such that if $\mathcal{L}$ is any recursively enumerable set of $m \times n$ matrices over nonnegative integers, then there is a matrix $A$ such that the matrices in $\mathcal{L}$ are the nonnegative matrices in the set $\{AM^{m_1}NM^{m_2}N \cdots NM^{m_q}B \mid m_1, \ldots, m_q \geq 0\}$. We use this result to deduce an undecidability result for products of matrices which can be viewed as a variant of Rice's theorem stating that all nontrivial properties of recursively enumerable sets are undecidable.



## 1. Introduction

The purpose of this paper is to show that all recursively enumerable sets of matrices over nonnegative integers can be obtained by using simple products of matrices. More precisely, for any positive integers $m$ and $n$, we can compute three matrices $M$, $N$ and $B$ over nonnegative integers and a positive integer $q$ such that if $\mathcal{L}$ is any recursively enumerable set consisting of $m \times n$ matrices with nonnegative integer entries, then we can compute a matrix $A$ over integers such that the matrices in $\mathcal{L}$ are exactly the nonnegative matrices in the set

$$\{AM^{m_1}NM^{m_2}N \cdots NM^{m_q}B \mid m_1, \ldots, m_q \geq 0\}.$$

This result implies that already in a very simple setup the product of matrices with integer entries gives rise to as complex behavior as is theoretically possible. Indeed, because the product of matrices is a computable operation, it follows by Church's thesis (see [13]) that any set obtained as above is recursively enumerable.

The main idea in our proof is to use the universal Diophantine representation of recursively enumerable sets of nonnegative integers. The existence of such a representation was proved by Matiyasevich in connection of his undecidability proof for Hilbert's tenth problem, see [11]. The second idea is to use products of matrices to express the values taken by the universal polynomial. This trick is often used, for example, in the study of decision problems concerning rational series in noncommuting variables (see [9,14]).

We will apply this representation result to prove a general undecidability result for products of matrices. This result can be viewed as a variant of Rice's well-known theorem in the theory of recursive functions stating that every nontrivial property of recursively enumerable sets is undecidable.

We assume that the reader is familiar with the basics concerning recursively enumerable sets (see [13]). Intuitively, a set is recursively enumerable if there is an effective procedure to list the elements of the set. Formally, a set is recursively enumerable if there is a Turing machine accepting the set.

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There is a large number of papers dealing with various decidability questions concerning matrices. The inspiration for this paper comes largely from [1], where the authors use Hilbert's tenth problem to prove that it is undecidable whether or not certain sets of products of given square matrices contain the zero matrix. Other important papers concerning this topic include, but are certainly not limited to, the papers [12,10,7,8,3,6,2] and [4].

## 2. The results

As usual, $\mathbb{N}$ and $\mathbb{Z}$ are the sets of nonnegative integers and all integers. Suppose $m$ and $n$ are positive integers and $R$ is a semiring. Then the set of $m \times n$ matrices having entries in $R$ is denoted by $R^{m\times n}$ and the set of upper-triangular $m \times m$ matrices is denoted by $\mathrm{Tri}(m, R)$. If $1 \le i \le m$ and $1 \le j \le n$, then $E(i, j)$ is the $m \times n$ matrix whose only nonzero entry is 1 at position $(i, j)$. (In what follows the size of $E(i, j)$ is not specified if it is clear from the context.)

Now, fix an enumeration

$$\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots$$

of the recursively enumerable subsets of $\mathbb{N}^{m\times n}$. In the next section we will prove the following result.

**Theorem 1.** *Let $m, n \ge 1$. We can compute positive integers $p, q$ and matrices $A \in \mathbb{Z}^{m\times p}$, $M, N \in \mathrm{Tri}(p, \mathbb{N})$ and $B \in \mathbb{N}^{p\times n}$ such that*

$$\mathcal{L}_s = \left\{AM^sNM^{m_1}NM^{m_2}N\cdots NM^{m_q}B \;\middle|\; m_1,\ldots,m_q \ge 0\right\} \cap \mathbb{N}^{m\times n}$$

*for all $s \ge 0$.*

Let $A, M, N$ and $B$ be as in Theorem 1. By Theorem 1, if $\mathcal{L}$ is any recursively enumerable subset of $\mathbb{N}^{m\times n}$ then there is a matrix $A_1$ such that the matrices belonging to $\mathcal{L}$ are the nonnegative matrices in the set

$$\left\{A_1M^{m_1}NM^{m_2}N\cdots NM^{m_q}B \;\middle|\; m_1,\ldots,m_q \ge 0\right\}.$$

Here $A_1 = AM^sN$ where $s$ is chosen so that $\mathcal{L} = \mathcal{L}_s$. Observe that if we change the set $\mathcal{L}$ we only have to change the matrix $A_1$. The same matrices $M, N, B$ work for all $\mathcal{L}$.

**Example 1.** Let $m, n \ge 1$ and let $M, N, B$ and $q$ be as in Theorem 1. Then we can compute a matrix $A_2$ such that if we take the products

$$A_2M^{m_1}NM^{m_2}N\cdots NM^{m_q}B$$

for $m_1,\ldots,m_q \ge 0$ which are nonnegative then we get exactly those $m \times n$ matrices for which all entries are prime numbers.

Similarly, we can compute a matrix $A_3$ such that we get all $m \times n$ matrices which have the property that the total number of distinct prime factors of the entries equals 2014.

**Example 2.** For this example we first observe that any book in any library can be regarded as a finite set of matrices in a natural and uniform way. To do this, assume that the book is printed so that each page contains at most 70 lines and each line contains at most 100 symbols. Here we regard the empty space, punctuation marks and so on as symbols. Next we replace each symbol by a positive integer chosen to represent the symbol. In this way each page can be viewed as a $70 \times 100$ matrix with positive integer entries. To get the finite set of matrices representing a given book we add to the matrices representing the pages of the book a new row whose first entry gives the page number while the remaining 99 entries equal zero.

Now Theorem 1 implies that we can compute positive integers $p$ and $q$ and matrices $A \in \mathbb{Z}^{71\times p}$, $M, N \in \mathrm{Tri}(p, \mathbb{N})$ and $B \in \mathbb{N}^{p\times 100}$ such that

$$\mathcal{L}_s = \left\{AM^sNM^{m_1}NM^{m_2}N\cdots NM^{m_q}B \;\middle|\; m_1,\ldots,m_q \ge 0\right\} \cap \mathbb{N}^{71\times 100}$$

where $\mathcal{L}_s$ is the $s$th recursively enumerable subset of $\mathbb{N}^{71\times 100}$ in a fixed enumeration of the recursively enumerable subsets of $\mathbb{N}^{71\times 100}$. It follows that by taking a suitable matrix $A_1$ the nonnegative matrices in

$$\left\{A_1M^{m_1}NM^{m_2}N\cdots NM^{m_q}B \;\middle|\; m_1,\ldots,m_q \ge 0\right\}$$

give us Hamlet. By using a different $A_1$ we get the Bible and so on. The same matrices $M, N$ and $B$ work for all existing and forthcoming books.

The use of nonsquare matrices in Theorem 1 cannot be avoided. Indeed, if $A, M, N, B \in \mathbb{Z}^{m\times m}$ and $q \ge 1$, then the sets

$$\left\{AM^sNM^{m_1}NM^{m_2}N\cdots NM^{m_q}B \;\middle|\; m_1,\ldots,m_q \ge 0\right\} \cap \mathbb{N}^{m\times m}$$