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Parameterized complexity of connected even/odd subgraph problems [☆]

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ABSTRACT

In 2011, Cai an Yang initiated the systematic parameterized complexity study of the following set of problems around Eulerian graphs: for a given graph G and integer k, the task is to decide if G contains a (connected) subgraph with k vertices (edges) with all vertices of even (odd) degrees. They succeed to establish the parameterized complexity of all cases except two, when we ask about:

- a connected k-edge subgraph with all vertices of odd degrees, the problem known as k-Edge Connected Odd Subgraph; and
- a connected k-vertex induced subgraph with all vertices of even degrees, the problem known as k-Vertex Eulerian Subgraph.

We show that k-Edge Connected Odd Subgraph is FPT and k-Vertex Eulerian Subgraph is W[1]-hard. Our FPT algorithm is based on a novel combinatorial result on the treewidth of minimal connected odd graphs with even amount of edges.

1. Introduction

One of the oldest theorems in Graph Theory is attributed to Euler, and it says that a graph admits an Euler walk, i.e. a walk visiting every edge exactly once, if and only if the graph is connected and all its vertices are of even degrees. While checking if a given graph is Eulerian, i.e. is connected and has no vertices of odd degrees, is easily done in polynomial time, the problem of finding k edges in a graph to form an Eulerian subgraph is NP-hard. We refer to the book of Fleischner [7] for a thorough study of Eulerian graphs and related topics.

An *even* graph (respectively, *odd* graph) is a graph with each vertex of an even (odd) degree. Thus an Eulerian graph is a connected even graph. Let Π be one of the following four graph classes: Eulerian graphs, even graphs, odd graphs, and connected odd graphs. In [4], Cai and Yang initiated the study of parameterized complexity of subgraph problems motivated by Eulerian graphs. For each Π, they defined the following parameterized subgraph and induced subgraph problems:

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Table 1
Parameterized complexity of k-Edge Π Subgraph and k-Vertex Π Subgraph.

	Eulerian	Even	Odd	Connected Odd
k-Edge	FPT [4]	FPT [4]	FPT [4]	FPT Theorem 2
k-Vertex	W[1]-hard Theorem 3	FPT [4]	FPT [4]	FPT [4]

k-Edge Π Subgraph (resp. k-Vertex Π Subgraph)

Instance: A graph G and non-negative integer k.
Parameter: k.
Question: Does G contain a subgraph with k edges from Π (resp. an induced subgraph on k vertices from Π)?

Cai and Yang established the parameterized complexity of all variants of the problem except k-Edge Connected Odd Subgraph and k-Vertex Eulerian Subgraph, see Table 1. It was conjectured that k-Edge Connected Odd Subgraph is FPT and k-Vertex Eulerian Subgraph is W[1]-hard. We resolve these open problems and confirm both conjectures.

The remaining part of the paper is organized as follows. In Section 2, we provide definitions and give preliminary results. In Section 3, we show that k-Edge Connected Odd Subgraph is FPT. Our algorithmic result is based on an upper bound for the treewidth of a minimal connected odd graphs with an even number of edges. We show that the treewidth of such graphs is always at most 3. The proof of this combinatorial result, which we find interesting in its own, is non-trivial and is given in Section 4. The bound on the treewidth is tight—complete graph on four vertices K_4 is a minimal connected odd graph with an even number of edges and its treewidth is 3. In Section 5, we prove that k-Vertex Eulerian Subgraph is W[1]-hard and observe that the problem remains W[1]-hard if we ask about (not necessarily induced) Eulerian subgraph on k vertices. We conclude the paper in Section 6 with some open problems.

2. Definitions and preliminary results

Graphs. We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. A set $S \subseteq V(G)$ of pairwise adjacent vertices is called a *clique*. For a vertex v, we denote by $N_G(v)$ its *(open) neighborhood*, that is, the set of vertices which are adjacent to v. The distance between two vertices $u, v \in V(G)$ (i.e., the length of the shortest (u, v)-path in the graph) is denoted by $\mathrm{dist}_G(u, v)$. For a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, the distance between v and S is $\mathrm{dist}_G(v, S) = \min\{\mathrm{dist}_G(v, u) | u \in S\}$. For a vertex v and a positive integer k, $N_G^{(k)}[v] = \{u \in V(G) \mid \mathrm{dist}_G(u, v) \leqslant k\}$. The *degree* of a vertex v is denoted by $d_G(v)$, and $\Delta(G)$ is the maximum degree of G. For a set of vertices $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and by $G - S$ we denote the graph obtained form G by the removal of all the vertices of S, i.e. the subgraph of G induced by $V(G) \setminus S$.

Parameterized complexity. Parameterized complexity is a two dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and another one is a parameter k. It is said that a problem is *fixed parameter tractable* (or FPT), if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f. One of basic assumptions of the Parameterized Complexity theory is the conjecture that the complexity class W[1] $\neq$ FPT, and it is unlikely that a W[1]-hard problem could be solved in FPT-time. We refer to the books of Downey and Fellows [6], Flum and Grohe [8], and Niedermeier [9] for detailed introductions to parameterized complexity.

Treewidth. A *tree decomposition* of a graph G is a pair (X, T) where T is a tree and $X = \{X_i \mid i \in V(T)\}$ is a collection of subsets (called *bags*) of $V(G)$ such that:

1. $\bigcup_{i \in V(T)} X_i = V(G)$,
2. for each edge $\{x, y\} \in E(G)$, $x, y \in X_i$ for some $i \in V(T)$, and
3. for each $x \in V(G)$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max_{i \in V(T)}\{|X_i| - 1\}$. The *treewidth* of a graph G (denoted as $\mathbf{tw}(G)$) is the minimum width over all tree decompositions of G.

Minimal odd graphs with even number of edges. Let r be a vertex of G. We assume that G is *rooted* in r. Let G be a connected odd graph with an even number of edges. We say that G is a *minimal* if G has no proper connected odd subgraph with an even number of edges containing r.

The importance of minimal odd subgraphs with even numbers of edges is crucial for our algorithm because of the following combinatorial result.

Theorem 1. *Let G be a minimal connected odd graph with an even number of edges with a root r. Then $\mathbf{tw}(G) \leqslant 3$.*