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Effective computation of immersion obstructions for unions of graph classes

Archontia C. Giannopoulou [*,1,2], Iosif Salem, Dimitris Zoros

Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR-15784, Athens, Greece

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ABSTRACT

In the final paper of the Graph Minors series Robertson and Seymour proved that graphs are well-quasi-ordered under the immersion ordering. A direct implication of this theorem is that each class of graphs that is closed under taking immersions can be fully characterized by forbidding a finite set of graphs (immersion obstruction set). However, as the proof of the well-quasi-ordering theorem is non-constructive, there is no generic procedure for computing such a set. Moreover, it remains an open issue to identify for which immersion-closed graph classes the computation of those sets can become effective. By adapting the tools that were introduced by Adler, Grohe and Kreutzer, for the effective computation of minor obstruction sets, we expand the horizon of computability to immersion obstruction sets. In particular, our results propagate the computability of immersion obstruction sets of immersion-closed graph classes to immersion obstruction sets of finite unions of immersion-closed graph classes.

1. Introduction

The development of the graph minor theory constitutes a vital part of modern Combinatorics. A lot of theorems that were proved and techniques that were introduced in its context, appear to be of crucial importance in Algorithmics and the theory of Parameterized Complexity as well as in Structural Graph Theory. Such examples are the Excluded Grid Theorem [30], the Structural Theorems in [28,31] and the Irrelevant Vertex Technique in [27]. (For examples of algorithmic applications, see [12,22].)

We say that a graph H is an immersion (minor) of a graph G, if we can obtain H from a subgraph of G by lifting (contracting) edges followed by a possibly empty sequence of isolated vertex deletions. (For detailed definitions, see Section 2.) While the minor ordering has been extensively studied throughout the last decades [1,9,27,28,30–33], the immersion ordering has only recently gained more attention [13,22,34]. One of the fundamental results that appeared in the last paper of the Graph Minors series was the proof of Nash–Williams' Conjecture, that is, the class of all graphs is well-quasi-ordered by the immersion ordering [33]. A direct corollary of these results is that a graph class $\mathcal{C}$, which is closed under taking immersions, can be characterized by a finite family $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$ of minimal, according to the immersion ordering, graphs that are not contained in $\mathcal{C}$ (called obstructions from now on). Furthermore, in [22], it was proven that there is an $O(|V(G)|^3)$ algorithm that decides whether a graph H is an immersion of a graph G (where the hidden constants depend only on H).

* Corresponding author.
E-mail addresses: arcgian@math.uoa.gr (A.C. Giannopoulou), ysalem@math.uoa.gr (I. Salem), dzoros@math.uoa.gr (D. Zoros).

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Thus, an immediate algorithmic implication of the finiteness of $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$ and the algorithm in [22], is that it can be decided in cubic time whether a graph belongs to $\mathcal{C}$ or not (by testing if the graph G contains any of the graphs in $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$ as an immersion). In other words, these two results imply that membership in an immersion-closed graph class can be decided in cubic time. Such graph classes are, for example, the class of graphs $\mathcal{E}_t$ that admit a proper edge-coloring of at most t colors such that for every two edges of the same color every path between them contains an edge of greater color, $t \in \mathbb{N}$, [5] and the class of graphs $\mathcal{W}_k$ whose carving-width is at most k, $k \in \mathbb{N}$, [4].

We would like to mention here that the same meta-algorithmic conclusion also holds for the minor ordering from the proofs in [28] and [29]. Moreover, this result, that is, the existence of a cubic time algorithm deciding the membership of a graph in a graph class that is closed under minors, broadened the perspectives towards the understanding of the NP-hard problems.

It was actually at that point that it became clear what seemed to be as "*different levels of hardness*" between these problems [6]. Notice for example, for the well-known k-Vertex Cover problem, that the class of graphs admitting a vertex cover of size at most k is closed under taking minors. Therefore, for every fixed k there is a cubic time algorithm deciding whether a graph has a vertex cover of size k. However, no similar result can be expected for the k-Coloring problem, as it is known to be NP-hard for every fixed $k \geqslant 3$. The observation of this gap in the time complexity of the NP-hard problems facilitated the development of the Parameterized Complexity Theory [14,19,26] by M. Fellows and R. Downey, which has proven to be a very powerful theory and has majorly advanced during the past decades (for example, see [2,7,8,11,12]).

Nevertheless, the aforementioned meta-algorithmic result for an immersion-closed graph class $\mathcal{C}$ assumes that the family $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$ is known. Of course, the finiteness of $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$ directly implies that this family of graphs is computable. However, such an algorithm for the computation of $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$ would also require a description of the family. Therefore, while the existence of an algorithm computing the immersion obstruction set of an immersion-closed graph class $\mathcal{C}$ is affirmed, the construction of such an algorithm is, in general, elusive. Moreover, as the proofs in [29] and [33] are non-constructive (see [20]), no generic algorithm is provided that allows us to identify these obstruction sets for every immersion-closed graph class. Furthermore, even for fixed graph classes, this task can be extremely challenging as such a set could contain many graphs [15] and no general upper bound on its cardinality is known other than its finiteness. The issue of the effective computability of obstruction sets for minors and immersions was raised by M. Fellows and M. Langston [17,18] and the challenges against computing obstruction sets soon became clear. In particular, in [18] M. Fellows and M. Langston showed that the problem of determining obstruction sets from machine descriptions of minor-closed graph classes is recursively unsolvable (which directly holds for the immersion ordering as well). Moreover, in [10] B. Courcelle, R. Downey and M. Fellows proved that the obstruction set of a minor-closed graph class cannot be computed from a description of the minor-closed graph class in Monadic Second Order Logic (MSO). Thus, a natural open problem is to identify the information that is needed for an immersion-closed graph class $\mathcal{C}$ in order to make it possible to effectively compute the obstruction set $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$.

Several methods have been proposed towards tackling the non-constructiveness of these sets (see, for example, [9,17]) and the problem of algorithmically identifying minor obstruction sets has been extensively studied [1,9,10,17,18,24]. In [9], it was proven that the obstruction set of a minor-closed graph class $\mathcal{F}$ which is the union of two minor-closed graph classes $\mathcal{F}_1$ and $\mathcal{F}_2$ whose obstruction sets are given can be computed under the assumption that there is at least one tree that does not belong to $\mathcal{F}_1 \cap \mathcal{F}_2$ and in [1] it was shown that the aforementioned assumption is not necessary.

In this paper, we initiate the study for computing immersion obstruction sets. In particular, we deal with the problem of computing $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{C})$ for families of graph classes $\mathcal{C}$ that are constructed by finite unions of immersion-closed graph classes. Observe that the union and the intersection of two immersion-closed graph classes are also immersion-closed, hence their obstruction sets are of finite size. It is also easy to see that, given the obstruction sets of two immersion-closed graph classes, the obstruction set of their intersection can be computed in a trivial way. However, notice that, in general, while the combination of a machine description of an immersion-closed graph class $\mathcal{F}$, that is, an algorithm deciding membership of a graph in an immersion-closed graph class $\mathcal{F}$, with an upper bound on the size of the forbidden graphs makes the computation of $\mathbf{obs}_{\leqslant_{\mathrm{im}}}(\mathcal{F})$ effective, neither the machine description of the class nor the upper bound alone are sufficient information. Moreover, as mentioned before, no generic procedure is known for computing such an upper bound. Thus, the problem of computing the obstruction set of the union of two immersion-closed graph classes is not trivial.

We prove that there is an algorithm that, given the obstruction sets of two immersion-closed graph classes, outputs the obstruction set of their union. Our approach is based on the derivation of a uniform upper bound on the tree-width of the subgraph-minimal graphs that do not belong to an immersion-closed graph class $\mathcal{C}$. We then build on the machinery introduced by I. Adler, M. Grohe and S. Kreutzer in [1] for computing minor obstruction sets. In particular, we will ask for an MSO-description of an immersion-closed graph class $\mathcal{C}$ (instead of a machine description) and an upper bound on the tree-width of the subgraph-minimal graphs that contain an obstruction of $\mathcal{C}$ (instead of an upper bound on the size of the obstructions of $\mathcal{C}$).

For this, we adapt the results on [1] as to permit the computation of the obstruction set of any immersion-closed graph class $\mathcal{C}$, under the conditions that an explicit upper bound on the tree-width of the subgraph-minimal graphs that do not belong to $\mathcal{C}$ (and thus contain at least one of its obstructions) can also be computed and the graph class $\mathcal{C}$ can be defined in MSO. We present this algorithm in Lemma 4, and with that we conclude the computability part of the paper. Our next step is a combinatorial result proving a uniform upper bound on the tree-width of the subgraph-minimal graphs that do not belong to the union of two immersion-closed graph classes, whose obstruction sets are known. We then show that the