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Self-similar solutions for the diffraction of weak shocks

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ABSTRACT

We numerically solve a problem for the unsteady transonic small disturbance equations that describes the diffraction of a weak shock into an expansion wave. In the context of a shock moving into a semi-infinite wall, this problem describes the interaction between the reflected part of the shock and the part that is transmitted beyond the wall. We formulate the equations in self-similar variables, and obtain numerical solutions using high resolution finite difference schemes. Our solutions appear to show that the shock dies out at the sonic line, rather than forms at an interior point of the supersonic region.

1. Introduction

In classical numerical simulations of steady transonic flow over an airfoil, the shock which terminates the supersonic region appears to form exactly on the sonic line (see, for example, [1,3,5–7]). However, high-resolution numerical calculations done on this problem and a corresponding situation in pseudo-steady two-dimensional flow in [8] show that the shock actually forms at a point where the underlying system (steady or self-similar) is strictly hyperbolic, and the flow is supersonic. The calculations in [8] show that the shock forms when compressive characteristics reflect off the sonic line and converge inside the supersonic, hyperbolic region. This mechanism of shock formation had originally been proposed by Guderley in [2].

The numerical experiments in [8] provide a direct observation of Guderley's proposed supersonic shock formation. Those experiments suggest that shock formation generically occurs inside the supersonic region and away from the sonic line, although it may be possible in specific situations to cause the characteristics to converge exactly on the sonic line.

In this paper we numerically study a problem related to the formation of a shock described above: it is the *disappearance* of a shock that diffracts self-similarly into an expansion wave. We express the governing equations in self-similar variables, and we solve the self-similar equations using high-resolution schemes. We obtain solutions that are highly refined in the area of the shock disappearance point. In view of the numerical solutions for shock formation in [8], our solutions for shock disappearance are perhaps surprising: they appear to show that the shock disappears exactly on the sonic line, and not at a supersonic point.

We now describe the physical basis for the shock diffraction problem we solve here, which was introduced in [4]. Consider a weak plane shock moving at Mach number M slightly larger than 1 into a semi-infinite wall, as in Fig. 1. This problem is self-similar, so the solution depends only on $(x/t, y/t)$, where x and y are spatial coordinates and t is time. The incident shock diffracts around the wall, and an expansion wave (dotted line) propagates behind it. The reflected shock also diffracts around the wall, meeting the diffracted expansion at the point S. At S, there is a continuous transition from shock wavefront to expansion wavefront, and one of the main questions concerning this transition

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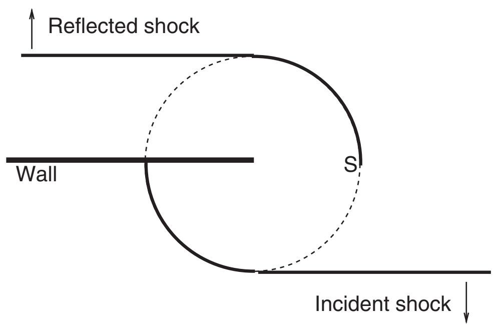

Fig. 1. A weak shock moving downwards at slightly supersonic velocity, into and past a semi-infinite wall. Solid lines are shocks; dotted lines are expansion waves.

point is whether or not it is sonic. It is the solution near S that we seek.

An asymptotic matching analysis, described in [4], shows that the solution near a point such as S is given in the weak shock limit by the solution of an "inner" shock-diffraction problem for the unsteady transonic small disturbance equations (UTSDE). This inner problem consists of the UTSDE together with matching data, and is given in Section 2 (Eqs. (2.1) and (2.2)). In this paper, we present numerical solutions of this inner shock-diffraction problem for the UTSDE computed in self-similar coordinates. There are several advantages to using our self-similar formulation rather than the standard unsteady formulation of the equations. First, solving the problem in self-similar form enables local grid refinement procedures to be implemented relatively easily, because solutions of the self-similar equations are stationary. As well, a grid continuation procedure is possible, whereby solutions are partially converged on coarse grids, interpolated onto more refined grids, and ultimately fully converged on the most refined grid. Finally, numerical evidence [9] suggests that shocks are captured more sharply in numerical solutions of the self-similar equations than in solutions of the unsteady equations. We use extreme local grid refinement in order to determine the nature of the flow (sonic or supersonic) at the shock disappearance point S. Our motivation for solving a problem for the UTSDE valid only near the shock disappearance point, rather than solving a problem for the full Euler equations on the global domain indicated in Fig. 1, is that, for the same computational cost, we can obtain a much more finely resolved solution near the area of interest – the shock disappearance point – than we could using the Euler equations.

Preliminary numerical solutions of the shock-diffraction problem for the UTSDE were given in [4], and appeared to indicate that the shock disappears on the sonic line. The highly refined solutions presented in this paper provide further evidence that the shock disappearance point is exactly sonic (see Fig. 5, for example). In addition, we give here a detailed description of the numerical method, including a discussion of its stability. In this paper, we also describe the effect of using different (linearized and "nonlinearized") far field numerical boundary conditions, and we compare and contrast our numerical solutions for the self-similar diffraction of a shock into an expansion wave with solutions for shock formation in steady and pseudo-steady transonic flows.

The organization of the rest of the paper is as follows. In Section 2, we describe the shock-diffraction problem for the UTSDE, and in Section 3 we outline the numerical method used to solve this problem. In Section 4 we present our numerical solutions, which provide evidence that the shock disappearance point is a sonic point. In Section 5, we discuss questions raised by these solutions, and compare them to the numerical solutions for the formation of shocks in [8]. In Section 6 we summarize our conclusions.

2. The shock-diffraction problem for the UTSDE

The shock-diffraction problem for the UTSDE consists of the UTSDE,

$$\begin{aligned} &u_t + \left(\frac{1}{2}u^2\right)_x + v_y = 0,\\ &u_y - v_x = 0, \end{aligned} \tag{2.1}$$

in $t > 0$, with the initial, or matching, condition

$$\begin{aligned} &u \sim \alpha \frac{y}{t}\sqrt{-\left(\frac{x}{t} + \frac{y^2}{4t^2}\right)} \quad \text{as } \frac{x}{t} + \frac{y^2}{4t^2} \to -\infty,\\ &u = 0 \quad \text{for } \frac{x}{t} + \frac{y^2}{4t^2} \text{ sufficiently large and positive.} \end{aligned} \tag{2.2}$$

Here, the variables $u(x, y)$ and $v(x, y)$ are proportional to the x and y velocity components, respectively. The small positive parameter α is proportional to the strength of the wave. Solutions of (2.1) and (2.2) asymptotically describe, in the weak shock limit, solutions of the full Euler equations near a point where a shock propagates into a constant state and diffracts into an expansion wave. See [4] for details of the formulation of this problem.

The problem (2.1) and (2.2) is self-similar, so its solution depends only on $\xi = x/t$ and $\eta = y/t$. Writing (2.1) in terms of ξ and η we get

$$\begin{aligned} -\xi u_\xi - \eta u_\eta + \left(\frac{1}{2}u^2\right)_\xi + v_\eta &= 0,\\ u_\eta - v_\xi &= 0. \end{aligned} \tag{2.3}$$

The initial condition (2.2) in the unsteady problem (2.1) and (2.2) becomes a far field boundary condition in the corresponding self-similar formulation of the problem. Writing (2.2) in terms of ξ and η gives

$$u \sim \alpha\eta\sqrt{-\left(\xi + \frac{\eta^2}{4}\right)}, \qquad v \sim -\frac{2}{3}\alpha\left(-\left(\xi + \frac{\eta^2}{4}\right)\right)^{3/2}$$

$$\text{as } \xi + \frac{\eta^2}{4} \to -\infty. \tag{2.4}$$

The far field behavior of v given in (2.4) follows from the far field behavior of u and the second equation in (2.3). We also have from (2.2) that

$$u = 0, \qquad v = 0 \quad \text{for } \xi + \frac{\eta^2}{4} \text{ sufficiently large and positive.} \tag{2.5}$$

Eqs. (2.3)–(2.5) are a formulation of the shock-diffraction problem in self-similar form. The data in boundary condition (2.4) exactly satisfies the linearization of (2.3). However, this data is continuous and has a square-root singularity at the wavefront, which in the linearized approximation is located at $\xi + \eta^2/4 = 0$. This is qualitatively incorrect since, at the wavefront, the solution of the nonlinear problem has a discontinuity across a shock and a finite jump in its derivative across an expansion. The following "nonlinearized" far field boundary data, which has the correct qualitative behavior at the wavefront (located in the nonlinearized approximation at $\xi + \eta^2/4 = 0$ for $\eta < 0$ and $\xi + \eta^2/4 = 3\alpha^2\eta^2/4$ for $\eta > 0$), was obtained in [4]:

$$\begin{aligned} &u \sim \alpha\eta\sqrt{-\left(\xi + \frac{\eta^2}{4}\right) + \alpha^2\eta^2} + \alpha^2\eta^2,\\ &v \sim -\frac{2}{3}\alpha\left(\sqrt{-\left(\xi + \frac{\eta^2}{4}\right) + \alpha^2\eta^2} + \alpha\eta\right)^3. \end{aligned} \tag{2.6}$$