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Independence friendly logic with classical negation via flattening is a second-order logic with weak dependencies

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ABSTRACT

It is well-known that Independence Friendly (IF) logic is equivalent to existential second-order logic (Σ_1^1) and, therefore, is not closed under classical negation. The Boolean closure of IF sentences, called Extended IF-logic, on the other hand, corresponds to a proper fragment of Δ_2^1. In this article we consider SL(↓), IF-logic extended with Hodges' flattening operator ↓, which allows to define a classical negation. SL(↓) contains Extended IF-logic and hence it is at least as expressive as the Boolean closure of Σ_1^1. We prove that SL(↓) corresponds to a weak syntactic fragment of SO which we show to be strictly contained in Δ_2^1. The separation is derived almost trivially from the fact that Σ_n^1 defines its own truth-predicate. We finally show that SL(↓) is equivalent to the logic of Henkin quantifiers, which shows, we argue, that Hodges' notion of negation is adequate.

1. Introduction

Independence Friendly logic (IF, for short), introduced by Hintikka and Sandu [1] and which became part of Hintikka's foundational programme for mathematics [2], is an extension of first-order logic (FO) where each disjunction and each existential quantifier may be decorated with denotations of universally quantified variables, as in

$$\forall x \forall y \exists z_{|\forall y} \exists w_{|\forall y} [y \approx z \vee_{|\forall x, \forall y} w \approx y]. \tag{1}$$

The standard interpretation of IF is through a variation of the classical game-theoretical semantics for FO: Eloïse's strategy function for a position of the form $\exists x_{|\forall y, \forall z} \psi$ or $\psi \vee_{|\forall y, \forall z} \chi$, under valuation v, cannot depend on neither $v(y)$ nor $v(z)$. Thus, we say that a sentence φ is *true* in model $\mathcal{A}$ (notation, $\mathcal{A} \models^+ \varphi$) if Eloïse has a winning strategy on the associated game; and that it is *false* (notation, $\mathcal{A} \models^- \varphi$) whenever Abélard has a winning strategy.

Now, the fact that Eloïse's strategy may not take into account all the available information turns the game into one of imperfect information. Thus, certain formula-structure pairs may have a non-determined semantic game; that is, one in which neither of the players has a winning strategy. As an example of non-determinacy, consider this formula:

$$\chi_1 := \forall x \exists y_{|\forall x} x \not\approx y. \tag{2}$$

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It is not hard to see that if $\mathcal{A}$ is a model with at least two elements, then $\mathcal{A} \not\models^{+} \chi_1$ and $\mathcal{A} \not\models^{-} \chi_1$. One says that χ_1 is neither *true* nor *false* in $\mathcal{A}$.

In game-theoretical semantics, negation is interpreted as a switch of roles, i.e., Abélard plays on Eloïse's former positions and vice versa. We use $\sim$ to denote this form of negation and we refer to it as *game negation*. For any IF-formula ψ and any model $\mathcal{A}$, $\mathcal{A} \models^{+} \psi$ iff $\mathcal{A} \models^{-} \sim\psi$ (i.e., Eloïse has a winning strategy for ψ on $\mathcal{A}$ iff Abélard has one for $\sim\psi$ on $\mathcal{A}$). However, observe that $\psi \vee \sim\psi$ is not in general a valid IF-formula (e.g., take ψ to be χ_1 in (2)). This means that game negation in IF is not equivalent to *classical negation*, which will be denoted with $\neg$ and is characterized by

$$\mathcal{A} \models^{+} \neg\psi \quad \text{iff} \quad \mathcal{A} \not\models^{+} \psi. \tag{3}$$

Since the expressive power of IF corresponds to that of existential second-order logic (Σ_1^1) [2,3] and Σ_1^1 is not closed under (classical) negation, it is clear that classical negation cannot be defined in IF.

Classical negation plays an important role in Hintikka's original programme. In [2], he claims that "virtually all of classical mathematics can in principle be done in extended IF first-order logic" (in a way that is ultimately "reducible" to plain IF logic). What he calls "(truth-functionally) extended IF logic" is the closure of the set of IF-sentences with operators $\neg$, $\wedge$ and $\vee$. Clearly, extended IF logic corresponds in expressive power to the Boolean closure of Σ_1^1, which is known to be a proper fragment of Δ_2^1 [4,5].

Hodges [6] shows that IF logic admits a Tarski-style compositional semantics and then extends his presentation to account also for extended IF. To support classical negation, he introduces the *flattening* operator $\downarrow$, which "restores two-valued logic on sentences" [6, p. 556]. That is, extended IF is obtained, roughly speaking, by considering the formulas where $\downarrow$ only occurs on certain positions (roughly speaking, $\sim\downarrow$ can occur where $\neg$ would occur in extended IF logic, see below). But because $\downarrow$ is given a compositional semantics, the logic where it is allowed to occur anywhere in a formula is well-defined. The natural question to ask is what is the logic one thus obtains, and this is the main topic of this paper.

One might suspect the resulting logic to be extremely expressive: freely combining classical negation with second order existential quantifiers leads to full second-order logic (SO). We will show that this is not the case: IF with unrestricted classical negation, in Hodges' style, corresponds to a rather mild fragment of SO, which is properly contained in Δ_2^1. This will be the subject of Section 4. The separation from Δ_2^1 is based on known results on truth-definitions for the analytical hierarchy [7,8] that, for the sake of completeness, are presented in Section 6.

Hodges' overall presentation is based on a mild extension of IF, called *slash logic* (SL), in which independence restrictions can occur in any connective (instead of only on $\exists$ and $\vee$). The unique feature of his *compositional* semantics is that the free variables are interpreted using a *set of variable assignments* (called *deals*), instead of just a variable assignment as in usual Tarski-style semantics for FO. In his terminology, a *trump* for a given game is a non-empty set of deals, V, such that some uniform strategy for Eloïse is winning for every game starting with any $v \in V$. To support classical negation, he extends slash logic with the flattening operator $\downarrow$. If we denote a set of variable assignments with V, its semantics can be given by

$$\mathcal{A} \models^{+} \downarrow\varphi[V] \quad \text{iff} \quad \mathcal{A} \models^{+} \varphi\big[\{v\}\big] \quad \text{for all } v \in V; \tag{4}$$

$$\mathcal{A} \models^{-} \downarrow\varphi[V] \quad \text{iff} \quad \mathcal{A} \not\models^{+} \varphi\big[\{v\}\big] \quad \text{for all } v \in V. \tag{5}$$

Then one defines $\neg\varphi$ as $\sim\downarrow\varphi$ and it is easy to verify that when restricted to formulas evaluated under a set composed of a single assignment $\{v\}$ (we omit the braces for readability), negation behaves as expected:

$$\mathcal{A} \models^{+} \neg\varphi[v] \quad \text{iff} \quad \mathcal{A} \models^{+} \sim\downarrow\varphi[v] \quad \text{iff} \quad \mathcal{A} \models^{-} \downarrow\varphi[v] \quad \text{iff} \quad \mathcal{A} \not\models^{+} \varphi[v]; \tag{6}$$

$$\mathcal{A} \models^{-} \neg\varphi[v] \quad \text{iff} \quad \mathcal{A} \models^{-} \sim\downarrow\varphi[v] \quad \text{iff} \quad \mathcal{A} \models^{+} \downarrow\varphi[v] \quad \text{iff} \quad \mathcal{A} \models^{+} \varphi[v]. \tag{7}$$

It is worth stressing out that the asymmetry in clauses (4) and (5), which in turns reflects in the asymmetry in (6) and (7) is fine. For instance, if in (5) the $\not\models^{+}$ were replaced by $\models^{-}$ then one would have that $\neg$ behaves exactly as $\sim$. Observe also that the semantics of $\downarrow$ is biased towards *falsity*: if a sentence φ in SL is neither true nor false then $\downarrow\varphi$ is false. Thus, when working with $\downarrow$, the adequate notion to study is being *true* ($\models^{+}$) vs. *not true* ($\not\models^{+}$) instead of being *true* vs. *false*. This is why we will study only the notion $\models^{+}$ in the context of SL with the operator $\downarrow$.

Hodges' slash logic with flattening (SL($\downarrow$)) admits a more convenient *second-order game* semantics, in which Abélard and Eloïse play what can be regarded as strategy functions for the standard game for SL. This will be the topic of Section 2; for a proof of the equivalence with the original compositional semantics, the reader is referred to [9].

Arguably, it could be possible that the semantics given to the flattening operator only made sense when restricted to sentences. Put in other words, it is not clear a priori that Hodges' characterization of classical negation for IF is the correct one. We investigate this in Section 5; we will see that SL($\downarrow$) coincides with the logic of Henkin quantifiers. The latter can be seen as the closure by (classical) negation of the logic in which only one top-level Henkin quantifier can be used, which is known to be equivalent to IF.

Some of the results contained in the present paper appeared in [10].