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On the determinacy of concurrent games on event structures with infinite winning sets

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ABSTRACT

We consider nondeterministic concurrent games played on event structures and study their determinacy problem—the existence of winning strategies. It is known that when the winning conditions of the games are characterised by a collection of finite winning sets/plays, a restriction (called race-freedom) on the boards where the games are played guarantees determinacy. However the games may no longer be determined when the winning sets are infinite. This paper provides a study of concurrent games and nondeterministic winning strategies by analysing conditions that ensure determinacy when infinitely many events are played, that is, when the winning sets are infinite. The main result is a determinacy theorem for a class of games with a bounded concurrency property and infinite winning sets shown to be finitely decidable.

1. Introduction

One of the most fundamental questions when studying games with winning conditions is determinacy, that is, the existence of winning strategies (see [1–4] for some examples particularly relevant in informatics). It is well-known that even the simplest setting where two players, which we call Eve (∃) and Adam (∀) hereafter, play against each other (possibly concurrently) without taking turns leads to games where determinacy fails.

Here we study conditions for which nondeterministic concurrent games, i.e. games where the players are allowed to use nondeterministic concurrent strategies, are determined. We consider concurrent games played on event structures, a model of concurrent computation where the causal dependencies between the events of a system are modelled as partially ordered structures. This paper, in particular, studies concurrent games on event structures where the winning conditions allow winning sets/plays that are infinite.

Concurrent games [5] form a model of interactive behaviour where nondeterministic strategies are formalised as certain maps of event structures. This games model, as first introduced in [5], did not allow for the definition of winning conditions. In order to overcome this limitation, in [6], the initial concurrent games model was extended with winning conditions and a determinacy result was given for games that satisfy two properties: firstly, a structural condition, *race-freedom*, which prevents a player from interfering with the moves available to the other; secondly, a restriction to winning conditions where only *finite* winning sets are allowed. This paper extends the work on concurrent games, mainly, by providing a new determinacy result.

The paper starts with a very general study of properties of concurrent games and strategies; in particular, operations on concurrent games which preserve the existence of nondeterministic winning strategies and a study relating strategies seen

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as maps of events structures to strategies seen as certain kinds of closure operators on the boards where the games are played.

Then, the main result of the paper is presented, namely a determinacy theorem for concurrent games where the winning conditions allow infinite winning sets. The theorem holds on a class of games which satisfies two properties: firstly, a structural property called *bounded concurrency* which ensures that no event (or move) of either player is concurrent with infinitely many events of the other player; and secondly, a restriction to concurrent games where the winning conditions determine infinite winning sets/plays (or winning configurations in the terminology of event structures) which can be regarded as the elements of a class of *concurrent open sets*. Such sets, which due to bounded concurrency can be shown to be finitely decidable, are a characteristic feature of the winning conditions of our concurrent games.

Structure of the paper: Sections 2 and 3 introduce event structures and concurrent games on event structures with winning conditions.[1] Sections 4 to 6 contain the main contributions of the paper, as described above. Finally, Section 7 presents related work, conclusions, and ideas for future work.

2. Preliminaries

Here we introduce event structures and concurrent games on them. The material in this preliminary section can be found in [6,5].

2.1. Event structures

An *event structure* $(E, \leqslant, \mathrm{Con})$ comprises a set E of *events* which are partially ordered by $\leqslant$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\{e' \mid e' \leqslant e\} \text{ is finite for all } e \in E,$$
$$\{e\} \in \mathrm{Con} \quad \text{for all } e \in E,$$
$$Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}$$
$$X \in \mathrm{Con} \,\&\, e \leqslant e' \in X \implies X \cup \{e\} \in \mathrm{Con}.$$

The *configurations*, $\mathcal{C}(E)$, of E consist of those subsets $x \subseteq E$ that are

Consistent: $\forall X \subseteq x.\ X \text{ is finite} \Rightarrow X \in \mathrm{Con}$, and

Down-closed: $\forall e, e'.\ e' \leqslant e \in x \implies e' \in x$.

Write $\mathcal{C}^{\omega}(E)$ for the set of *finite* configurations of E and $\mathcal{C}^{\infty}(E)$ for the set of *infinite* configurations of E. Two events which are both consistent and incomparable with respect to $\leqslant$ are regarded as *concurrent*.

Notation 1. We use $\rightarrowtriangle$ for the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leqslant e'$ and no event in between; also, we write $e\ co\ e'$ when e and e' are concurrent. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leqslant e'\}$, the down-closure of X; note that if $X \in \mathrm{Con}$ then $[X]$ is a configuration; in particular, for singletons we write $[e]$ instead of $[\{e\}]$. We also write $[e)$ for the configuration $[e] \setminus \{e\}$ which contains the finite set of events that e depends on. Moreover, we write $x \stackrel{e}{\longrightarrow\!\!\!\!\subset} x'$ if $x \cup \{e\} = x'$ or simply $x \longrightarrow\!\!\!\!\subset x'$ if e is irrelevant. Finally we often refer to an event structure $(E, \leqslant, \mathrm{Con})$ by referring to its set of events E and may use subscripts, e.g. as in $\leqslant_E$ or Con_E, when necessary.

Maps of event structures. Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \to E'$ such that for all $x \in \mathcal{C}(E)$ its direct image fx is in $\mathcal{C}(E')$ and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined)}, \quad \text{then } e_1 = e_2.$$

Partial maps of event structures compose as partial functions, with identity maps given by identity functions. For any event e a map $f : E \to E'$ must send the configuration $[e]$ to the configuration $f[e]$. A map is *total* if f is total. A total map f is *locally injective* in the sense that with respect to any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective.

Event structures are rich in useful constructions. For instance, the category of event structures has products and pullbacks (both forms of synchronised composition) and coproducts (nondeterministic sums). Also, event structures support a restriction operation and a simple form of hiding, called projection, associated with a factorisation system. Both such operations are defined next. Let $(E, \leqslant, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E \downarrow V =_{\mathrm{def}} (V, \leqslant_V, \mathrm{Con}_V)$, where $v \leqslant_V v'$ if $v \leqslant v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ if $X \in \mathrm{Con}$ & $X \subseteq V$. Consider a partial map of event structures $f : E \to E'$. Let $V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}$. Then f clearly factors into the

[1] Some of the notations used in this paper differ from the ones used in [6,5], where concurrent games are also studied. Instead, we sometimes use the terminology used in [7]; the differences are insubstantial with respect to the work being presented.