Journal of Computer and System Sciences 80 (2014) 1138–1151

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

Multiset rewriting over Fibonacci and Tribonacci numbers

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ARTICLE INFO

Article history:
Received 16 February 2012
Received in revised form 7 July 2013
Accepted 3 February 2014
Available online 4 April 2014

Keywords:
Multiset rewriting
Church–Rosser property
Confluence
Termination
Integer partitions
Partition identities
Fibonacci numbers
Tribonacci numbers
k-Step Fibonacci numbers

ABSTRACT

We show how techniques from the formal logic, can be applied directly to the problems studied completely independently in the world of combinatorics, the theory of integer partitions. We characterize equinumerous partition ideals in terms of the minimal elements generating the complementary order filters. Here we apply a general rewriting methodology to the case of filters having overlapping minimal elements. In addition to a 'bijective proof' for Zeckendorf-like theorems – that every positive integer is uniquely representable within the Fibonacci, Tribonacci and *k*-Bonacci numeration systems, we establish 'bijective proofs' for a new series of partition identities related to Fibonacci, Tribonacci and *k*-step Fibonacci numbers. The main result is proved with the help of a multiset rewriting system such that the system itself and the system consisting of its reverse rewriting rules, both have the Church–Rosser property, which provides an explicit bijection between partitions of two different types (represented by the two normal forms).

1. Motivating examples and summary

One basic activity in combinatorics is to establish partition identities by so-called 'bijective proofs', which amounts to constructing explicit bijections between two classes of the partitions at hand.

Forming an interdisciplinary bridge between Theoretical Computer Science and Combinatorics, the aim of this paper is to show how techniques from the formal logic world can be applied directly to specific problems studied completely independently in the theory of integer partitions.

The basic algebraic idea, *which allows us to clarify the problem*, is to characterize equinumerous partition ideals in terms of the minimal elements generating the order filters, the complements to the original ideals.

The *novelty* of our approach to the combinatorics of partitions is in the use of rewriting techniques (*two-directional* in the sense that forward and backward applications of rewrite rules head respectively for two different normal forms) for the purpose of establishing *explicit relevant bijections* between partitions of two different types (represented by these normal forms).

The case where the minimal elements of each of the order filters mentioned above are disjoint is fully covered by a general approach developed in Kanovich [8,9].

The aim of this paper is to address the challenging case of filters having overlapping minimal elements.

An inspiring example is the Fibonacci numeration system:

Theorem 1.1. *(See Zeckendorf [17].) Every positive integer is uniquely representable as a sum of distinct Fibonacci numbers, but where no two consecutive Fibonacci numbers are used.*

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http://dx.doi.org/10.1016/j.jcss.2014.04.006

In Section 3 we use the two-directional rewriting technique to construct a 'bijective proof' for Zeckendorf's theorem (see also [10]).

In Section 4 we establish 'bijective proofs' for a new series of partition identities related to Fibonacci, Tribonacci and k-Bonacci numbers with different initial values.

As a corollary, in Section 5 we justify Zeckendorf-like numeration systems based on Lucas, Tribonacci and k-Bonacci sequences.

2. Backgrounds

Let us recall the background material with which we are dealing (see, for instance, Andrews [1]).

2.1. Integer partitions

An *integer partition* of n: $n = m_1 + m_2 + \cdots + m_k$, can be identified as a *multiset* M consisting of positive integers $m_1, m_2, \ldots, m_k$. We will represent this M as: $M = \{m_1, m_2, \ldots, m_k\}$, where the number of copies of some m shows the multiplicity of the m within M. Each m_i is called a *part* of the partition. This sum $m_1 + m_2 + \cdots + m_k$ will be also denoted by $\|M\|$:

$$\|M\| = m_1 + m_2 + \cdots + m_k. \tag{1}$$

Definition 2.1. Two classes of partitions $\mathcal{C}_1$ and $\mathcal{C}_2$ are called *equinumerous*, if for all n:

$$p(\mathcal{C}_1, n) = p(\mathcal{C}_2, n).$$

Here $p(\mathcal{C}, n)$ stands for the number of partitions of n that belong to a given class $\mathcal{C}$.

Example 2.1. Zeckendorf's theorem reads that: *For any positive integer n, the number of partitions of n into ones equals the number of partitions of n into distinct Fibonacci parts with no two consecutive Fibonacci parts (both numbers are equal to* 1*).*

Definition 2.2. (See Andrews [1].) Generally, the classes $\mathcal{C}$ of partitions considered in the literature have the 'local' property that if M is a partition in $\mathcal{C}$ and one or more parts are removed from M to form a new partition M', then M' is also in $\mathcal{C}$. Such a class $\mathcal{C}$ is called a *partition ideal*, or an *order ideal* in terms of the lattice $\mathcal{P}$ of finite multisets of positive integers, ordered by $\subseteq$. We say that $M' \subseteq M$ if M' can be formed by removing a number of parts from M. E.g., $\{1,5\} \subseteq \{1,1,3,5,5,5\}$.

Dually, a class $\mathcal{F} \subseteq \mathcal{P}$ is an *order filter* if $M' \in \mathcal{F}$, whenever $M \in \mathcal{F}$ and $M \subseteq M'$.

It is readily seen that $\mathcal{C}$ is a partition ideal if and only if its complement $\bar{\mathcal{C}}$ is an order filter.

Definition 2.3. M is *minimal* in an order filter $\mathcal{F}$, if $M \in \mathcal{F}$ and $M' = M$, for any multiset $M' \in \mathcal{F}$ such that $M' \subseteq M$. The *support* of $\mathcal{F}$ is defined as the set of all its minimal elements.

The case where the minimal elements of the corresponding order filters are *disjoint* is covered by the following general criterion from Kanovich [8].

Theorem 2.1. *(See Kanovich [8].) Let $\mathcal{C}$ and $\mathcal{C}'$ be partition ideals such that the support of the filter $\bar{\mathcal{C}}$ (the filter which is the complement to $\mathcal{C}$) is made of pairwise disjoint multisets, say $C_0, C_1, \ldots, C_i, \ldots$, and the support of the filter $\overline{\mathcal{C}'}$ (the filter which is the complement to $\mathcal{C}'$) is made of pairwise disjoint multisets $C'_0, C'_1, \ldots, C'_i, \ldots$.*

Assume that these supports are sorted as lists so that the sequence of integers $\|C_0\|, \|C_1\|, \ldots, \|C_i\|, \ldots$ is non-decreasing, and the sequence of integers $\|C'_0\|, \|C'_1\|, \ldots, \|C'_i\|, \ldots$ is non-decreasing.

Then $\mathcal{C}$ and $\mathcal{C}'$ are equinumerous if and only if:

$$\|C_i\| = \|C'_i\|, \quad \text{for all } i.$$

On top of that, the system Γ consisting of the following multiset rewriting rules:

$$\gamma_0 : C_0 \to C'_0, \quad \gamma_1 : C_1 \to C'_1, \quad \ldots, \quad \gamma_i : C_i \to C'_i, \quad \ldots$$

provides a bijection h between $\mathcal{C}'$ and $\mathcal{C}$ – meaning that, for any n, the function h introduced by the formula (7) in Definition 2.4 with this Γ, is a bijection between the partitions of n that belong to $\mathcal{C}'$ and the partitions of the n that belong to $\mathcal{C}$.

In its turn, the inverse bijection h^{-1} from $\mathcal{C}$ onto $\mathcal{C}'$ can be computed along the lines of Definition 2.4 with the system Γ^{-1} consisting of the 'reverse' rewriting rules:

$$\gamma_0^{-1} : C'_0 \to C_0, \quad \gamma_1^{-1} : C'_1 \to C_1, \quad \ldots, \quad \gamma_i^{-1} : C'_i \to C_i, \quad \ldots$$