Journal of Computer and System Sciences 77 (2011) 379–392

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

Theory and application of width bounded geometric separators ☆

Bin Fu

Department of Computer Science, University of Texas–Pan American, Edinburg, TX 78539, USA

ARTICLE INFO

Article history:
Received 30 October 2004
Received in revised form 18 February 2008
Available online 20 May 2010

Keywords:
Width bounded separator
Disk covering
Divide and conquer

ABSTRACT

We introduce the notion of the width bounded geometric separator and develop the techniques for the existence of the width bounded separator in any fixed d-dimensional Euclidean space. The separator is applied in obtaining $2^{O(\sqrt{n})}$ time exact algorithms for a class of NP-complete geometric problems, whose previous algorithms take $n^{O(\sqrt{n})}$ time. One of those problems is the well-known disk covering problem, which seeks to determine the minimal number of fixed size disks to cover n points on a plane. They also include some NP-hard problems on disk graphs such as the maximum independent set problem, the vertex cover problem, and the minimum dominating set problem. For a constant $a > 0$ and a set of points Q on the plane, an a-wide separator is the region between two parallel lines of distance a that partitions Q into Q_1 (on the left side of the region), S (inside the region), and Q_2 (on the right side of the region). If the distance is at least one between every two points in the set Q with n points, there is an a-wide separator that partitions Q into Q_1, S and Q_2 such that $|Q_1|, |Q_2| \leqslant (2/3)n$, and $|S| \leqslant 1.2126a\sqrt{n}$. Our width bounded separator in d-dimensional Euclidean space with fixed d is controlled by two parallel hyper-planes, and is used to design $2^{O(n^{1-\frac{1}{d}})}$-time algorithms for the d-dimensional disk covering problem and the above other problems in the d-dimensional disk graphs.

1. Introduction

The geometric separator has applications in many problems [27,8,7,35]. It plays an important role in the divide and conquer algorithms for geometric problems. Lipton and Tarjan [26] proved a well-known separator theorem for planar graphs. They proved that every n vertices planar graph has at most $\sqrt{8n}$ vertices whose removal separates the graph into two disconnected parts of size at most $\frac{2}{3}n$. Their $\frac{2}{3}$-separator was improved to $\sqrt{6n}$ by Djidjev [10], $\sqrt{5n}$ by Gazit [16], $\sqrt{4.5n}$ by Alon, Seymour and Thomas [32] and $1.97\sqrt{n}$ by Djidjev and Venkatesan [11]. Spielman and Teng [37] proved that for every planar graph, there exists a separator that has size at most $1.82\sqrt{n}$ and partitions the graph into two parts with size at most $\frac{3}{4}n$ in each side. Separators for more general graphs were derived in (e.g., [17,6,34]).

Some other forms of the geometric separators were studied by Miller, Teng, Thurston, and Vavasis [31,30] and Smith and Wormald [36]. For a set of points on the plane, assume each point is covered by a regular geometric object such as circle, rectangle, etc. If every point on the plane is covered by at most k objects, it is called k-thick. Some $O(\sqrt{k \cdot n})$ size separators and their algorithms were derived in [31,30,36].

The planar graph separators were applied in deriving some $2^{O(\sqrt{n})}$-time algorithms for certain NP-hard problems on planar graphs by Lipton, Tarjan [27], Ravi and Hunt [35]. Those problems include computing the maximum independence

☆ This research is supported by the Louisiana Board of Regents fund under contract number LEQSF(2004-07)-RD-A-35 and National Science Foundation Early Career Award 0845376.

E-mail address: binfu@cs.panam.edu.

0022-0000/$ – see front matter
doi:10.1016/j.jcss.2010.05.003

set, minimum vertex covers and three-colorings of a planar graph, and the number of satisfying truth assignments to a planar 3CNF formula [25]. Smith and Wormald [36] applied their separators in deriving $n^{O(\sqrt{n})}$-time algorithms for some geometric problems such as the planar Traveling Salesman and Steiner Tree problems. The separators were applied to the parameterized independent set problem on planar graphs by Alber, Fernau and Niedermeier [3,4] and disk graphs by Alber and Fiala [5].

We introduce the concept of width bounded separator. For a set of points Q on the plane, an a-wide separator is the region between two parallel lines of distance a. It partitions the set Q into two balanced subsets and its size is measured by the number of points of Q in the strip region. Our width bounded separator concept is geometrically natural and can achieve a much smaller constant c for its size upper bound $c\sqrt{n}$ than the previously approaches. Fu and Wang [15] developed a method for deriving a separator theorem for grid points by controlling the distance to the separator line. They proved that for a set of n grid points on the plane, there is a separator that has at most $1.129\sqrt{n}$ points and has at most $\frac{2}{3}n$ points on each side. They applied their separator to derive the first sub-exponential time algorithm for the protein folding problem in the HP model. This paper not only generalizes the results of [15], but also substantially improves the techniques in [15]. We now summarize our technical results. (1) In order to apply the separator to more general geometric problems with arbitrary input points other than grid points, we use weighted points in Euclid space and the sum of weights to measure the separator size instead of counting the number of points close to it. We introduce the local binding method to merge some points into a nearby grid point. This method is combined with our separator in deriving $2^{O(\sqrt{n})}$ time exact algorithms for a class of NP-complete geometric problems, whose previous algorithms take $n^{O(\sqrt{n})}$ time [2,5,1]. One of those problems is the well-known disk covering problem, which seeks to determine the minimal number of fixed size disks to cover n points on a plane [13]. They also include some NP-hard problems on disk graphs such as the maximum independent set problem, the vertex cover problem, and minimum dominating set problem. (2) We will handle the case of higher dimension. We develop an area ratio method to replace the previous angle ratio method [15] for obtaining higher dimensional separators. (3) We develop a similar separator theorem for a set of points with distance of at least 1 between any two of them. We establish the connection between this problem and the famous fixed size disks packing problem. The disks packing problem in 2D was solved in the combinatorial geometry (see [33]). The 3D case, which is the Kepler conjecture, has a very long proof (see [20]). It is still a very elusive problem at higher dimensions. Our Theorem 15 shows how the separator size depends on packing density. (4) We develop a simple polynomial time algorithm to find the width-bounded separator for a fixed dimensional space. This is a starting point for the algorithms finding the width bounded geometric separator, and it is enough to obtain the $2^{O(\sqrt{n})}$ time exact algorithms for a class of NP-hard geometric problems.

In Section 3, we prove some existence theorems for width bounded separator. In Section 4, we give a polynomial time algorithm for finding a width bounded separator. In Section 5, we describe the $2^{O(\sqrt{n})}$-time algorithm for the disk covering problem and the maximum independent set problem on disk graphs using the results in Section 3.

2. Overview of our methods

We describe our techniques in the 2-dimensional case. For a set of arbitrary points Q on the plane, we also consider another set of grid points P on the plane. Each point $p \in P$ is assigned a positive weight used to measure the density for the points of set Q near p. A good separator line L partitions the set Q into two balanced parts. Furthermore, it is also expected to have a small number of points from Q close to it. This will be measured by the sum of the weights of the points of P close to L.

A point p, which is usually a grid point, in P is combined with its weight to control the local region nearby p. The local region near a point p may contain a lot of points from Q and is assigned a weight depending on the number of points of Q close to p. We use small number of different weights by taking the integer part of the logarithm on the number of points close each point $p \in P$. This strategy is motivated by reducing the exponent of computational time complexity when the separator is applied in the dividing and conquer algorithms for the disk covering problem and other problems in the disk graphs.

The point set Q has a center point o (see Lemma 1) such that every line through it has a balanced partition for Q. For a random line L through the center point, the expected sum of the weights of points P close to it is a maximum when all points of P stay at the smallest circle with center at o. Furthermore, the points of P with larger weights are closer to the center than the points with smaller weights. When it has the small number of different weights, we can easily compute such an expectation, which is used as the upper bound for the quality of the best separator.

Finding an a-wide separator is straightforward with $O(n^3)$-time. Let each point $p \in P$ be covered with an insulation circle with radius $a/2$ and center at p. A good separator line L can be moved until it touches points of Q or until it is tangent to some insulation circles with radius $a/2$ and center at points of P. The movement neither crosses any point of Q nor enters any new insulation circle of P. It does not change the balance result and size measure of the separator. Trying all lines that either pass through points of Q or are tangent to the insulation circles with center at points of P, the separator can be obtained in $O(n^3)$ time.

To cover a set of points Q on the plane, the set P is a set of grid points such that each point in Q is close to at least one point in P. A grid point p is assigned the weight i if there are 2^i to 2^{i+1} points of Q on the 1×1 grid square with p