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On the maximum independent set problem in subclasses of subcubic graphs ☆

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ABSTRACT

It is known that the maximum independent set problem is NP-complete for subcubic graphs, i.e. graphs of vertex degree at most 3. Moreover, the problem is NP-complete for 3-regular Hamiltonian graphs and for H-free subcubic graphs whenever H contains a connected component which is not a tree with at most 3 leaves. We show that if every connected component of H is a tree with at most 3 leaves and at most 7 vertices, then the problem can be solved for H-free subcubic graphs in polynomial time. We also strengthen the NP-completeness of the problem on 3-regular Hamiltonian graphs by showing that the problem is APX-complete in this class.

1. Introduction

In a graph, an *independent set* is a subset of vertices no two of which are adjacent. The maximum independent set problem (MaxIS for short) consists in finding in a graph an independent set of maximum cardinality. This problem is generally NP-complete [6]. Moreover, it remains NP-complete even under substantial restrictions, for instance, for planar graphs or graphs of large girth [12]. On the other hand, for graphs in some particular classes, such as perfect graphs or claw-free graphs [11], the problem can be solved in polynomial time. In order to better understand the boundary between the NP-complete and polynomially-solvable cases of the problem, in the present paper we study MaxIS restricted to graphs of vertex degree at most 3 (also known as subcubic graphs), which is the best possible restriction expressed in terms of vertex degree under which the problem remains NP-complete. This restriction can also be expressed in terms of forbidden induced subgraphs, in which case the set of excluded graphs consists of 11 minimal graphs containing a vertex of degree 4. However, in terms of forbidden induced subgraphs the restriction to subcubic graphs is not best possible, because the problem is NP-complete in the class of $(K_{1,4}, K_3)$-free graphs, which is a proper subclass of subcubic graphs. This follows, in particular, from the result in [1] that can be stated as follows: if Z is a *finite* set containing no graph every connected component of which is a tree with at most three leaves, then MaxIS is NP-complete in the class of Z-free graphs. In other words, for polynomial-time solvability of the problem in a class defined by *finitely many* forbidden induced subgraphs, we must exclude a graph every connected component of which has the form $S_{i,j,k}$ represented in Fig. 1. Whether this condition is sufficient for polynomial-time solvability of the problem is a big open question.

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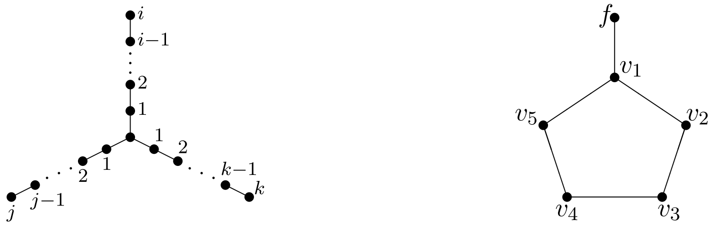

Fig. 1. Graphs $S_{i,j,k}$ (left) and A_5 (right).

Without the restriction on vertex degree, polynomial-time solvability of the problem in classes of $S_{i,j,k}$-free graphs was shown only for very small values of i, j, k. In particular, the problem can be solved for $S_{1,1,1}$-free (claw-free) graphs [11], $S_{1,1,2}$-free (fork-free) graphs [8], and $S_{0,1,1} + S_{0,1,1}$-free ($2P_3$-free) graphs [10]. Recently, Lokshtanov, Vatshelle, and Villanger [7] proved that the independence number of an $S_{0,2,2}$-free (P_5-free) graph can be computed in polynomial time (thereby solving a long-standing open problem).

With the restriction on vertex degree, we can do much better. In particular, we can solve the problem for P_k-free graphs of degree at most d for any k and d, because under this restriction the number of vertices in *connected* graphs is bounded by a function of k and d. More generally, we can solve the problem for $S_{1,j,k}$-free graphs of bounded degree for any j and k, because by excluding $S_{1,j,k}$ we exclude large apples (see definition in the end of the introduction), and for graphs of bounded degree containing no large apples the problem can be solved in polynomial time, which was recently shown in [9]. However, nothing is known about the complexity of the problem in classes of $S_{i,j,k}$-free graphs of bounded degree where all three indices i, j, k are at least 2. To make a progress in this direction, we consider best possible restrictions of this type, i.e. we study $S_{2,2,2}$-free graphs of vertex degree at most 3, and show that the problem is solvable in polynomial time in this class. More generally, we show that the problem is polynomial-time solvable in the class of H-free subcubic graphs whenever H is a graph every connected component of which is isomorphic to $S_{2,2,2}$ or to $S_{1,j,k}$. This result is presented in Section 2.

In Section 3, we switch to the classes where the problem is difficult and prove a new result in this area. In particular, we show that MaxIS is APX-complete in the class of 3-regular Hamiltonian graphs, which strengthens the NP-completeness of the problem in this class.

Section 4 concludes the paper with a number of open problems.

All graphs in this paper are simple, i.e. undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, we denote by $N(v)$ the neighborhood of v, i.e., the set of vertices adjacent to v, and by $N[v]$ the closed neighborhood of v, i.e. $N[v] = N(v) \cup \{v\}$. For $v, w \in V(G)$, we set $N[v, w] = N[v] \cup N[w]$. The *degree* of v is the number of its neighbors, i.e., $d(v) = |N(v)|$. The subgraph of G induced by a set $U \subseteq V(G)$ is obtained from G by deleting the vertices outside of U and is denoted $G[U]$. If no induced subgraph of G is isomorphic to a graph H, then we say that G is H-free. Otherwise we say that G contains H. If G contains H, we denote by $[H]$ the subgraph of G induced by the vertices of H and all their neighbors. As usual, by C_p we denote a chordless cycle of length p. Also, an *apple* A_p, $p \ge 4$, is a graph consisting of a cycle C_p and a vertex f which has exactly one neighbor on the cycle. We call vertex f the *stem* of the apple. See Fig. 1 for the apple A_5. The size of a maximum independent set in G is called the *independence number* of G and is denoted $\alpha(G)$.

2. Polynomial-time results

In this section, we show that the problem is polynomial-time solvable in the class of H-free subcubic graphs whenever H is a graph every connected component of which is isomorphic to $S_{2,2,2}$ or to $S_{1,j,k}$. We start by solving the problem for $S_{2,2,2}$-free subcubic graphs. To this end, we quote the following result from [9].

Theorem 2.1. *For any positive integers d and p,* MaxIS *is polynomial-time solvable in the class of* $(A_p, A_{p+1}, \ldots)$*-free graphs with maximum vertex degree at most d.*

We solve MaxIS for $S_{2,2,2}$-free subcubic graphs by reducing it to subcubic graphs without large apples.

Throughout the section we let G be an $S_{2,2,2}$-free subcubic graph and $K \ge 1$ a large fixed integer. If G contains no apple A_p with $p \ge K$, then the problem can be solved for G by Theorem 2.1. Therefore, from now on we assume that G contains an induced apple A_p with $p \ge K$ formed by a chordless cycle $C = C_p$ of length p and a stem f. We denote the vertices of C by $v_1, \ldots, v_p$ (listed along the cycle) and assume without loss of generality that the only neighbor of f on C is v_1 (see Fig. 1 for an illustration).

If v_1 is the only neighbor of f in G, then the deletion of v_1 together with f reduces the independence number of G by exactly 1. This can be easily seen and also is a special case of a more general reduction described in Section 2.1. The deletion of f and v_1 destroys the apple A_p. The idea of our algorithm is to destroy all large apples by means of other