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Approximating vertex cover in dense hypergraphs

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ABSTRACT

We consider the Minimum Vertex Cover problem in hypergraphs in which every hyperedge has size k (also known as *Minimum Hitting Set* problem, or *minimum set cover* with element frequency k). Simple algorithms exist that provide k-approximations, and this is believed to be the best approximation achievable in polynomial time. We show how to exploit density and regularity properties of the input hypergraph to break this barrier. In particular, we provide a randomized polynomial-time algorithm with approximation factor $k/(1+(k-1)\frac{\bar{d}}{k\Delta})$, where $\bar{d}$ and Δ are the average and maximum degree, and Δ must be $\Omega(n^{k-1}/\log n)$. The proposed algorithm generalizes the recursive sampling technique of Imamura and Iwama (SODA'05) for vertex cover in dense graphs. As a corollary, we obtain an approximation factor arbitrarily close to $k/(2-1/k)$ for subdense regular hypergraphs, which is shown to be the best possible under the Unique Games conjecture.

1. Introduction

A *vertex cover* of a hypergraph is a subset of its vertices hitting every hyperedge. The *Minimum Vertex Cover* problem in hypergraphs, also known as *minimum hitting set*, consists of finding a vertex cover of minimum size in a given hypergraph, and is among the most fundamental problems in combinatorial optimization. The decision version is a classical NP-complete problem. In this contribution, we will concentrate on the special case in which the input hypergraph is *k-uniform*: every hyperedge contains exactly k vertices. The Minimum Vertex Cover problem in k-uniform hypergraphs is equivalent to the *Set Cover* problem where each element of the universe occurs in exactly k sets. For $k=2$, it is the classical Vertex Cover problem in graphs. A simple 2-approximation algorithm exists for this problem by constructing a maximal matching greedily. However, currently best known approximation algorithms can only achieve an approximation ratio of $2-o(1)$ [10,14]. The problem is k-approximable in k-uniform hypergraphs, by choosing a maximal set of nonintersecting edges, and picking all vertices in them. The best known approximation factor is $k-(k-1)\ln\ln n/\ln n$ and is due to Halperin [10].

On the inapproximability side, one of the first hardness result for Vertex Cover in k-uniform hypergraphs is due to Trevisan [23]. He obtained an inapproximability factor of $k^{1/19}$. Holmerin [11] proved that it is NP-hard to approximate within $k^{1-\epsilon}$, and in addition, in [12], that the Minimum Vertex Cover problem in 4-uniform hypergraphs is NP-hard to approximate within $2-\epsilon$. Dinur et al. proved a $(k-3-\epsilon)$ lower bound [6], later improved to $(k-1-\epsilon)$ [7]. In [18],

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Khot introduced the Unique Games Conjecture (UGC) as an approach to tackle inapproximability. Assuming the UGC, an inapproximability factor of $k-\epsilon$ is due to Khot and Regev [19]. Recently, Bansal and Khot [3] were able to show that this UGC-based inapproximability bound also holds on k-uniform hypergraphs that are almost k-partite.

For the Vertex Cover problem on k-partite k-uniform hypergraphs, Lovász [21] achieved a $k/2$ approximation by rounding the natural LP relaxation. In [1], a tight integrality gap of $k/2-o(1)$ was given for the LP relaxation. Guruswami and Saket [9] recently showed it is NP-hard to approximate the Minimum Vertex Cover problem on k-partite k-uniform hypergraphs to within a factor of $k/4-\epsilon$, and within a factor of $k/2-\epsilon$ assuming the UGC for $k \geqslant 3$.

In order to break these complexity barriers, we propose to consider dense instances of the Vertex Cover problem in hypergraphs, as was done previously for many other problems [2,16,17,15]. Typically, a graph is said to be dense whenever the number of edges is within a constant factor of n^2, where n is the number of vertices. The Vertex Cover problem in dense hypergraphs, defined as hypergraphs with $\Omega(\binom{n}{k})$ hyperedges, has been considered before by Bar-Yehuda and Kehat [4]. They proposed an approximation algorithm which achieves a better approximation ratio than k and showed that it is best possible under some assumptions similar to the UGC. To the authors' knowledge, this is the only result tackling the dense version of the Vertex Cover problem in hypergraphs.

However, dense instances of the Vertex Cover problem in graphs ($k=2$) have been considered previously by Karpinski and Zelikovsky [17], Eremeev [8], Clementi and Trevisan [5], and later by Imamura and Iwama [13]. In the latter, the authors give an approximation ratio parameterized by both the average and the maximum degree of the graph, that is strictly smaller than 2 whenever the ratio between the two is bounded, and the graph has average degree $\Omega(n \cdot \log\log n/\log n)$. Other special instances of the Vertex Cover problem in hypergraphs are studied in [20,22].

1.1. Definitions

We use the notation $[i] := \{1, 2, \dots, i\}$. Let S be a finite set and $k \in [|S|]$, we introduce the abbreviation $\binom{S}{k}$ for the set of all $S' \subseteq S$ such that $|S'| = k$. A *k-uniform hypergraph* is a pair (V, E), where V is the vertex set, and E is a subset of $\binom{V}{k}$. We will usually set $n := |V|$ and $m := |E|$. In the remainder, unless stated explicitly, we will suppose that $k = O(1)$. A *Vertex Cover* of a k-uniform hypergraph (V, E) is a set $C \subseteq V$ such that $e \cap C \neq \emptyset$ $\forall e \in E$. The Minimum Vertex Cover problem consists of finding a vertex cover of minimum size in a given hypergraph.

The *degree* $d(v)$ of a vertex v is equal to $|\{e \in E\colon\ v \in e\}|$. Analogously, we define the degree $d(S)$ of an $S \subseteq V$ to be $|\{e \in E\colon\ S \subseteq e\}|$. We refer to the average and maximum degree of a vertex as $\bar{d}$ and Δ, respectively. A hypergraph is said to be *d-regular* whenever $d(v) = d$ $\forall v \in V$, and *regular* whenever there exists a d such that it is d-regular. A k-uniform hypergraph V is said to be *k-partite* whenever V can be partitioned in k classes $V_1, \dots, V_k$, so that $|e \cap V_i| = 1$ $\forall e \in E$, $i \in [k]$.

We say that a k-uniform hypergraph is *ℓ-wise ϵ-dense* for $\ell + 1 \in [k]$ and $\epsilon \in [0, 1]$ whenever for every subset $S \in \binom{V}{\ell}$, we have $d(S) \geqslant \epsilon\binom{n-\ell}{k-\ell}$. Thus, for instance a 0-wise ϵ-dense k-uniform hypergraph is a hypergraph with at least $\epsilon\binom{n}{k}$ hyperedges, while a 1-wise ϵ-dense k-uniform hypergraph is such that every vertex is contained in at least $\epsilon\binom{n-1}{k-1}$ hyperedges. This definition naturally generalizes the notion of *weak* and *strong density* in graphs [17]. For a given k-uniform hypergraph H, we define $\psi(n) := \binom{n}{k-1}/\Delta$ and call H *subdense* if $\bar{d} = \Omega(\frac{n^{k-1}}{\log n})$.

1.2. Our results

We propose new approximation algorithms for Vertex Cover in k-uniform hypergraphs with approximation factors parameterized by the density and regularity parameters of the input hypergraph, and prove their optimality assuming the UGC.

In 2004, Bar-Yehuda and Kehat [4] showed that the minimum vertex cover in 0-wise ϵ-dense k-uniform hypergraphs was approximable within a factor $k/(k-(k-1)(1-\epsilon)^{\frac{1}{k}}) + o(1)$. In Section 2, we generalize this result to ℓ-wise ϵ-dense hypergraphs with $\ell > 0$. Furthermore, we improve slightly on the approximation factor and provide a simpler proof. Our approach also allows us to introduce randomization in order to reduce the search space, which is useful for tackling the subdense case.

In Section 3, we propose a polynomial-time randomized algorithm yielding an approximation factor arbitrarily close to $k/(1+(k-1)\frac{\bar{d}}{k\Delta})$ on k-uniform hypergraphs with $\bar{d} = \Omega(n^{k-1}/\log n)$. This implies an approximation algorithm with approximation factor approaching $k/(2-1/k)$ for subdense regular hypergraphs. The proposed algorithm combines Imamura and Iwama's recursive sampling technique [13] for vertex cover in dense graphs (external recursion) with new ideas and the randomized approach for ϵ-dense hypergraphs (internal recursion). In particular, it extends the range of applicability compared to [13] by a factor of $\log\log n$ while still achieving the same approximation ratio. More precisely, we prove the following theorem:

Theorem 1. *For every $\epsilon > 0$, there is a randomized approximation algorithm which computes with high probability a solution for the Vertex Cover problem on k-uniform hypergraphs with approximation ratio*