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On the number of shortest descending paths on the surface of a convex terrain

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ABSTRACT

The shortest paths on the surface of a convex polyhedron can be grouped into equivalence classes according to the sequences of edges, consisting of n-triangular faces, that they cross. Mount (1990) [7] proved that the total number of such equivalence classes is $\Theta(n^4)$. In this paper, we consider descending paths on the surface of a 3D terrain. A path in a terrain is called a descending path if the z-coordinate of a point p never increases, if we move p along the path from the source to the target. More precisely, a *descending path* from a point s to another point t is a path Π such that for every pair of points $p = (x(p), y(p), z(p))$ and $q = (x(q), y(q), z(q))$ on Π, if $dist(s, p) < dist(s, q)$ then $z(p) \geqslant z(q)$. Here $dist(s, p)$ denotes the distance of p from s along Π. We show that the number of equivalence classes of the shortest descending paths on the surface of a convex terrain is $\Theta(n^4)$. We also discuss the difficulty of finding the number of equivalence classes on a convex polyhedron.

1. Introduction

A terrain $\mathcal{T}$ is a polyhedral surface in 3D with the property that the vertical line at any point on the xy-plane intersects the surface of $\mathcal{T}$ at most once. Thus, the projections of all the faces of a terrain on the xy-plane are pairwise disjoint in their interior. Each vertex p on the surface of the terrain is specified by a triple $(x(p), y(p), z(p))$. More formally, a terrain $\mathcal{T}$ is the image of a real bivariate function ζ defined on a compact and connected domain $\mathcal{W}$ in the Euclidean plane, i.e., $\mathcal{T} = \{(x, y, \zeta(x, y)),\ (x, y) \in \mathcal{W}\}$. Without loss of generality, we assume that all the faces of the terrain are triangles.

The problem of finding *descending paths* in a polyhedral terrain was first studied by de Berg and van Kreveld [5]. They presented an $O(n \log n)$ time algorithm to decide if there exists a descending path between two given points on the surface of the terrain, where n is the number of faces of the triangulated terrain. The problem of computing a *shortest descending path* (*SDP*) was first addressed by Roy et al. [8] who solved the problem for a convex terrain. Ahmed and Lubiw [3] gave an approximation algorithm for a terrain but in a restricted setting. Ahmed et al. [1] devised two approximation algorithms for a general terrain, both based on the idea of discretizing the terrain by adding Steiner points. Ahmed and Lubiw [2] explored the characteristics of bend angles along an SDP, and showed that finding an exact SDP in a general terrain is non-trivial.

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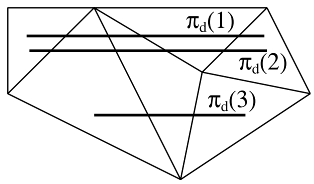

Fig. 1. Example of equivalence classes.

Ahmed and Lubiw [4] have recently extended this result and devised algorithms for generalizations of convex terrains and orthogonal terrains.

In this paper, we address the problem of finding the number of combinatorially different SDPs, or in other words, the number of *equivalence classes* of SDPs, in a convex terrain, where all the SDPs that use the same sequence of edges belong to the same equivalence class. This work is motivated by the result of Mount [7], who demonstrated that shortest paths on the surface of a convex polyhedron can be grouped into $\Theta(n^4)$ equivalence classes. Our aim here is to check whether this bound holds for SDPs. The main result of this paper is that the bound holds for SDPs on a convex terrain. We also mention the difficulties of finding such a bound for SDPs on a convex polyhedron.

2. Preliminaries

Let f and f' be two adjacent faces of the terrain $\mathcal{T}$ that share an edge e. Let p and q be two points on faces f and f' respectively (none of p and q is on e). Now, if the line segment $[p,q]$ is not visible (respectively visible) then f and f' are said to be in convex (respectively concave) position. A terrain $\mathcal{T}$ is said to be *convex* (resp. *concave*) if every two adjacent faces in $\mathcal{T}$ are in convex (resp. concave) position. Let us consider a convex terrain $\mathcal{T}$ with n vertices. Let $\pi_d(s,t)$ denote the (unique) shortest descending path through the surface of $\mathcal{T}$ between a pair of points s and t. Let $E(s,t)$ denote the sequence of terrain edges that the path $\pi_d(s,t)$ traverses. We will use the term *shortest descending edge sequence* (or simply *descending edge sequence*) to refer to $E(s,t)$. Two paths on the surface of $\mathcal{T}$ are said to be *equivalent* if they pass through the same edge sequence. For example in Fig. 1, paths $\pi_d(1)$ and $\pi_d(2)$ are equivalent but $\pi_d(3)$ is not equivalent to $\pi_d(1)$ and $\pi_d(2)$. Note that this notion of equivalence classes is similar to the equivalence classes of the geodesic paths on the surface of a convex polyhedron, proposed by Mount [7].

A path $\pi(s,t)$ from a point s to a point t on the surface of the terrain is said to be a geodesic path if it entirely lies on the surface of the terrain, it is locally optimal (i.e., the length of the path cannot be reduced by small perturbation), it is not self-intersecting, and its intersection with a face is either empty or a straight line segment. The geodesic distance $dist(p,q)$ between a pair of points p and q on $\pi(s,t)$ is the length of the path from p to q along $\pi(s,t)$. A path $\pi_{geo}(s,t)$ is said to be the geodesic shortest path if the distance between s and t along $\pi_{geo}(s,t)$ is minimum among all possible geodesic paths from s to t.

Definition 2.1. (See [6].) Let f and f' be a pair of faces in $\mathcal{T}$ that share an edge e. The *planar unfolding* of face f' with respect to face f is the image of the points of f' when rotated about the line e onto the plane containing f such that the points of f and the points of f' lie in two different sides of the edge e (i.e., faces f' and f do not overlap after unfolding).

Lemma 2.1. *(See [6].) For a pair of points s and t, if $\pi_{geo}(s,t)$ passes through an edge-sequence $\mathcal{E}$ of a terrain, then in the planar unfolding $U(\mathcal{E})$, $\pi_{geo}(s,t)$ is a straight line segment $[s^*,t^*]$, where s^* and t^* are the projections of s and t on $U(\mathcal{E})$.*

3. Properties of shortest descending paths on a convex terrain

We now discuss some important properties of the family of descending paths on a convex terrain that are useful to prove the main result of this paper.

Observation 1. *(See [8].) If p_1, p_3 are two points on a face of a terrain $\mathcal{T}$, and p_2 is another point on the line segment $[p_1,p_3]$, then $z(p_1) > z(p_2)$ implies $z(p_3) < z(p_2)$.*

Definition 3.1. A path $\pi(s,t)$ $(z(s) \geqslant z(t))$ on the surface of a terrain is a *descending path* if for every pair of points $p, q \in \pi(s,t)$, $dist(s,p) < dist(s,q)$ implies $z(p) \geqslant z(q)$.

From now onwards, we will use $\pi_d(s,t)$ and $\delta(s,t)$ to denote shortest descending path from s to t and its length, respectively. It is well known that, if a shortest descending path $\pi_d(s,t)$ passes through a face f, then the intersection of $\pi_d(s,t)$ with f is a straight line segment [8]. Furthermore, if the straight line segment $[s^*,t^*]$ corresponding to a shortest geodesic path $\pi_{geo}(s,t)$ satisfies the descending property in the unfolded plane $U(\mathcal{E})$, then $\pi_d(s,t) = \pi_{geo}(s,t)$.