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Space complexity of perfect matching in bounded genus bipartite graphs

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ABSTRACT

We investigate the space complexity of certain perfect matching problems over bipartite graphs embedded on surfaces of constant genus (orientable or non-orientable). We show that the problems of deciding whether such graphs have (1) a perfect matching or not and (2) a unique perfect matching or not, are in the log-space complexity class *Stoic Probabilistic Log-space* (SPL). Since SPL is contained in the log-space counting classes $\bigoplus$L (in fact in Mod_kL for all $k \geqslant 2$), $\mathrm{C}_=$L, and PL, our upper bound places the above-mentioned matching problems in these counting classes as well. We also show that the search version, computing a perfect matching, for this class of graphs can be performed by a log-space transducer with an SPL oracle. Our results extend the same upper bounds for these problems over bipartite planar graphs known earlier. As our main technical result, we design a log-space computable and polynomially bounded weight function which isolates a minimum weight perfect matching in bipartite graphs embedded on surfaces of constant genus. We use results from algebraic topology for proving the correctness of the weight function.

1. Introduction

The *perfect matching* problem and its variations are one of the most well-studied problems in theoretical computer science. Research in understanding the inherent complexity of computational problems related to matching has lead to important results and techniques in complexity theory and elsewhere in theoretical computer science. However, even after decades of research, the exact complexity of many problems related to matching is not yet completely understood.

We investigate the *space complexity* of certain well-studied perfect matching problems over bipartite graphs. We prove new uniform space complexity upper bounds on these problems for *graphs embedded on surfaces of constant genus*. We prove our upper bounds by solving the technical problem of 'deterministically isolating' a perfect matching for this class of graphs.

Distinguishing a single solution out of a set of solutions is a basic algorithmic problem with many applications. The *Isolation Lemma* due to Mulmuley, Vazirani, and Vazirani provides a general randomized solution to this problem. Let $\mathcal{F}$ be a non-empty set system on $U = \{1, \ldots, n\}$. The Isolation Lemma says, for a random weight function on U (bounded by $n^{O(1)}$), with high probability there is a *unique* set in $\mathcal{F}$ of minimum weight [1]. This lemma was originally used to give an elegant RNC algorithm for constructing a maximum matching (by isolating a minimum weight perfect matching) in general graphs. Since its discovery, the Isolation Lemma has found many applications, mostly in discovering new randomized or

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non-uniform upper bounds, via isolating minimum weight solutions [1–4]. Clearly, derandomizing the Isolation Lemma in sufficient generality will improve these upper bounds to their deterministic counterparts and hence will be a major result. Unfortunately, recently it is shown that such a derandomization will imply certain circuit lower bounds and hence is a difficult task [5].

Can we bypass the Isolation Lemma altogether and deterministically isolate minimum weight solutions in specific situations? Recent results illustrate that one may be able to use the structure of specific computational problems under consideration to achieve non-trivial deterministic isolation. In [6], the authors used the structure of directed paths in planar graphs to prescribe a simple weight function that is computable deterministically in logarithmic space with respect to which the minimum weight directed path between any two vertices is unique. In [7], the authors isolated a perfect matching in planar bipartite graphs. In [8], the authors give a generalized weight function that isolates directed paths in planar graphs and perfect matchings in undirected bipartite planar graphs by bypassing the use of grid graphs altogether. In this paper we extend the deterministic isolation technique of [7,8] to isolate a minimum weight perfect matching in bipartite graphs embedded on constant genus surfaces. This is more interesting in light of the fact that for constant genus graphs even the existence of a polynomially bounded weight function, that isolates a minimum weight perfect matching, was not known earlier. As a future direction it would be interesting to consider the general bipartite graph $K_{n,n}$, and prove the existence of a polynomially bounded weight function that isolates a minimum weight perfect matching in this case.

1.1. Our contribution

Let G be a bipartite graph and let $\overrightarrow{G}$ be the directed graph obtained by replacing every undirected edge $\{u, v\}$ of G with the directed edges (u, v) and (v, u). The main technical contribution of the present paper can then be stated (semi-formally) as follows.

- **Main technical result.** Given an embedding of an undirected bipartite constant genus graph G, there is a log-space matching preserving reduction f, and a log-space computable, polynomially bounded, skew-symmetric weight function w for the class of directed graphs, so that the weight of any simple cycle in $\overrightarrow{f(G)}$ with respect to w is non-zero.
  We use this result to establish (using known techniques) the following new upper bounds. Refer to the next section for definitions.
- **New upper bounds.** For bipartite graphs, combinatorially embedded on surfaces of constant genus, we show that (a) checking if the graph has a perfect matching is in SPL, (b) checking if the graph has a unique perfect matching is in SPL, and (c) constructing a perfect matching (if one exists) is in $\mathsf{FL}^{\mathsf{SPL}}$.

SPL is a log-space complexity class that was first studied by Allender, Reinhardt, and Zhou [4]. This is the class of problems reducible to the determinant with the promise that the determinant is either 0 or 1. In [4], the authors show, using a non-uniform version of Isolation Lemma, that perfect matching problem for general graphs is in a 'non-uniform' version of SPL. In [7], using the above-mentioned deterministic isolation, the authors show that for planar bipartite graphs, Decision-BPM is in fact in SPL (uniformly). Recently, Hoang showed that for graphs with polynomially many matchings, perfect matchings and many related matching problems are in SPL [9]. SPL is contained in log-space counting classes such as $\mathrm{Mod}_k\mathrm{L}$ for all $k \geqslant 2$ (in particular in $\bigoplus\mathrm{L}$), PL, and $\mathrm{C_{=}L}$, which are in turn contained in NC^2. PL and $\mathrm{C_{=}L}$ contain the class nondeterministic log-space or NL. But no relation is known between the classes $\mathrm{Mod}_k\mathrm{L}$ (for all $k \geqslant 2$), PL, and $\mathrm{C_{=}L}$. Thus the upper bound of SPL that we prove implies that the problems Decision-BPM and Unique-BPM for the class of graphs we study are in these log-space counting classes as well.

The techniques that we use in this paper can also be used to isolate directed paths in graphs on constant genus surfaces. This shows that the reachability problem for this class of graphs can be decided in the unambiguous class UL (a subclass of NL), extending the results of [6]. But this upper bound is already known since recently Kynčl and Vyskočil show that reachability for bounded genus graphs log-space reduces to reachability in planar graphs [10].

Matching problems over graphs of low genus have been of interest to researchers, mainly from a parallel complexity viewpoint. For the class of bipartite planar graphs, it was shown that finding a perfect matching can be done in NC [11]. In [12], the authors present an NC^2 algorithm for computing a perfect matching for bipartite graphs on surfaces of $O(\log n)$ genus (readers can also find an account of known parallel complexity upper bounds for matching problems over various classes of graphs in their paper). However, the space complexity of matching problems for graphs of low genus has not been investigated before. The present paper takes a step in this direction.

1.2. Proof outline

We assume that the graph G is presented as a combinatorial embedding on a surface (orientable or non-orientable) of genus g, where g is a constant. This is a standard assumption when dealing with graphs on surfaces, since it is NP-complete to check whether a graph has genus $\leqslant g$ [13]. We first give a sequence of two reductions to get, from G, a graph G' with an embedding on a genus g 'polygonal schema in normal form'. These two reductions work for both orientable and non-orientable cases. At this point we take care of the non-orientable case by reducing it to the orientable case. These reductions are matching preserving, bipartiteness preserving and computable in log-space. Finally, for $\overrightarrow{G'}$ (the directed version of G'),