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Power domination in certain chemical structures

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ABSTRACT

Let $G(V,E)$ be a simple connected graph. A set $S \subseteq V$ is a power dominating set (PDS) of G, if every vertex and every edge in the system is observed following the observation rules of power system monitoring. The minimum cardinality of a PDS of a graph G is the power domination number $\gamma_p(G)$. In this paper, we establish a fundamental result that would provide a lower bound for the power domination number of a graph. Further, we solve the power domination problem in polyphenylene dendrimers, Rhenium Trioxide (ReO_3) lattices and silicate networks.

1. Introduction

A dominating set of a graph $G(V,E)$ is a set S of vertices of G such that every vertex (node) in $V - S$ has at least one neighbor in S. The problem of finding a dominating set of minimum cardinality is an important problem that has been extensively studied. The minimum cardinality of a dominating set of G is its *domination number*, denoted by $\gamma(G)$. A variation called the power domination problem has been formulated as a graph domination problem by Haynes et al. in [13].

For a vertex v of G, let $N(v)$ and $N[v]$ denote the open and closed neighborhoods of v respectively. For a set S, let $N(S) = \bigcup_{v \in S} N(v) - S$ and $N[S] = N(S) \cup S$ denote the open and close neighborhoods of S respectively. For vertices $x, y \in V$, let the denotation $x \sim y$ mean that x is adjacent to y.

Let G be a connected graph and S a subset of its vertices. Then we denote the set observed by S with $M(S)$ and define it recursively as follows:

1. (domination)
   $M(S) \leftarrow S \cup N(S)$
2. (propagation)
   As long as there exists $v \in M(S)$ such that
   $N(v) \cap (V(G) - M(S)) = \{w\}$
   set $M(S) \leftarrow M(S) \cup \{w\}$

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A set S is called a power dominating set (PDS) of G if $M(S) = V(G)$. The power domination number $\gamma_p(G)$ is the minimum cardinality of a PDS of G. A PDS of G with the minimum cardinality is called a $\gamma_p(G)$-set. Since any dominating set is a power dominating set, $1 \le \gamma_p(G) \le \gamma(G)$ for all graphs G. We say a graph G is power dominated by a set S if all its vertices are observed.

Many chemical structures such as Sierpínski networks [21], silicate networks [18], tetrahedral diamond lattice [1] were modelled as graphs and studied. This motivated us to model polyphenylene dendrimers and ReO_3 lattice as graphs and as possible electrical power networks. This paper is divided into five sections. Section 2 deals with a brief literature survey and Section 3 deals with a fundamental result that would provide a lower bound for the power domination number of a graph. We call this result the power domination – subgraph relation. Sections 4, 5 and 6 deal with the power domination problem in polyphenylene dendrimers, ReO_3 lattices and silicate networks respectively. For terms not defined in the paper the reader may refer to [15].

2. Previous work

The problem of deciding if a graph G has a power dominating set of cardinality k has been shown to be NP-complete for bipartite graphs, chordal graphs [13] and split graphs [16]. The power domination problem has efficient polynomial time algorithms for the classes of trees [13], graphs with bounded treewidth [12], block graphs [24], block-cactus graphs [14], interval graphs [16], grids [20], honeycomb meshes [23] and circular-arc graphs [17]. Upper bounds on the power domination number are given for a connected graph with at least three vertices, for a connected claw-free cubic graph [25], for hypercubes [5], and for generalized Petersen graphs [3]. Closed formulae for the power domination number are obtained for Mycielskian of the complete graph, the wheel, the n-fan and n-star [22], for Cartesian product of paths and cycles [3,10], for tensor and strong product of paths with paths [9], and for tensor product of paths with cycles [22].

3. Power domination–subgraph relation

We begin this section with a fundamental result and illustrate its application by deducing a few existing theorems.

Theorem 3.1 *(Power domination–subgraph relation). Let $H_1, H_2, \ldots, H_k$ be pairwise disjoint subgraphs of G satisfying the following conditions*

1. *$V(H_i) = V_1(H_i) \cup V_2(H_i)$ where $V_1(H_i) = \{x \in V(H_i) | x \sim y$ for some $y \in V(G) - V(H_i)\}$ and $V_2(H_i) = \{x \in V(H_i) | x \nsim y$ for all $y \in V(G) - V(H_i)\}$.*
2. *$V_2(H_i) \neq \emptyset$ and for each $x \in V_1(H_i)$, there exist at least two vertices in $V_2(H_i)$ which are adjacent to x.*

If $V_1(H_i)$ is observed and if l_i is the minimum number of vertices required to observe $V(H_i)$, then $\gamma_p(G) \ge \sum_{i=1}^{k} l_i$.

Proof. We need to show that from each copy of the subgraph H_i in G, at least l_i vertices belong to any power dominating set D. Let us prove by the method of contradiction. Let us assume that the graph G is power dominated by the set D where $D \cap V(H_i) = \emptyset$ for some i. Now two cases arise:

1. $N(D) \cap V(H_i) \neq \emptyset$
2. $N(D) \cap V(H_i) = \emptyset$

In both these cases, vertices in $V_2(H_i)$ are not observed as every vertex in $V_1(H_i)$ has at least two vertices in $V_2(H_i)$ to which it is adjacent. Thus, the graph G is not power dominated, contradicting the assumption. Further other vertices cannot observe $V_2(H_i)$ as $V(H_i) \cap V(H_j) = \emptyset$, $i, j \in \{1, 2, \ldots, k\}$. Since $|N(D) \cap V(H_i)|$ is at most $|V_1(H_i)|$, l_i vertices must belong to D. As the argument holds true for all subgraphs H_i, $i \in \{1, 2, \ldots, k\}$, we have $\gamma_p(G) \ge \sum_{i=1}^{k} l_i$. □

Let us recall that a vertex in a tree adjacent to a leaf is called a *support* vertex and a vertex adjacent to two or more leaves is called a *strong support* vertex.

Theorem 3.2. *(See [13].) If v is a strong support vertex in a tree G, then v is in every $\gamma_p(G)$-set.*

Definition 3.3. An s^{th} complete binary tree $B(s)$ is a graph whose node set is $\{0, 1, 2, \ldots, 2^s - 2\}$ and edge set is $\{(i, j) | \lfloor \frac{j}{2} \rfloor = i\}$. A vertex v of a tree is said to be at level j if its distance from the root is $j - 1$. There are s levels in $B(s)$.

Theorem 3.4. *(See [13].) Let G be a complete binary tree of height h. Then $\gamma_p(G) = 2^{h-2}$.*

Incidently, Theorem 3.4 can be deduced from Theorem 3.1 by taking each H_i as $K_{1,2}$.