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Nearly tight approximation bounds for vertex cover on dense k-uniform k-partite hypergraphs

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ABSTRACT

We establish almost tight upper and lower approximation bounds for the Vertex Cover problem on dense k-uniform k-partite hypergraphs.

1. Introduction

A hypergraph $H = (V, E)$ consists of a vertex set V and a collection of hyperedges E where a hyperedge is a subset of V. H is called k-uniform if every edge in E contains exactly k vertices. A subset C of V is a vertex cover of H if every edge $e \in E$ contains at least a vertex of C.

The *Vertex Cover* problem in a k-uniform hypergraph H is the problem of computing a minimum cardinality vertex cover in H. It is well known that the problem is *NP*-hard even for $k = 2$ (cf. [15]). On the other hand, the simple greedy heuristic which chooses a maximal set of nonintersecting edges, and then outputs all vertices in those edges, gives a k-approximation algorithm for the Vertex Cover problem restricted to k-uniform hypergraphs. The best known approximation algorithm achieves a slightly better approximation ratio of $(1 - o(1))k$ and is due to Halperin [13].

On the intractability side, Trevisan [24] provided one of the first inapproximability results for the k-uniform vertex cover problem and obtained an inapproximability factor of $k^{\frac{1}{19}}$ assuming $P \neq NP$. In 2002, Holmerin [13] improved the factor to $k^{1-\epsilon}$. Dinur et al. [9,10] gave consecutively two lower bounds, first $(k - 3 - \epsilon)$ and later on $(k - 1 - \epsilon)$. Moreover, assuming Khot's Unique Games Conjecture (UGC) [19], Khot and Regev [20] proved an inapproximability factor of $k - \epsilon$ for the Vertex Cover problem in k-uniform hypergraphs. The same inapproximability threshold was also proved by Bansal and Khot [4] for

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the Vertex Cover problem in k-uniform hypergraphs which are almost k-partite, assuming the UGC. Under the UGC, Austrin, Khot and Safra [3] gave an optimal inapproximability result for the Vertex Cover problem in bounded degree graphs.

The Vertex Cover problem restricted to k-partite k-uniform hypergraphs, when the underlying partition is given, was studied by Lovász [22] who achieved a $k/2$-approximation. This approximation upper bound is obtained by rounding the natural LP relaxation of the problem. The above bound on the integrality gap was shown to be tight in [1]. As for the lower bounds, Guruswami and Saket [12] proved that it is NP-hard to approximate the Vertex Cover problem on k-partite k-uniform hypergraphs to within a factor of $k/4-\epsilon$ for $k \geq 5$. Assuming the Unique Games Conjecture, they also provided an inapproximability factor of $k/2-\epsilon$ for $k \geq 3$. More recently, Sachdeva and Saket [23] obtained a nearly optimal *NP*-hardness factor. More precisely, they proved that this problem is *NP*-hard to approximate with an approximation ratio less than $k/2-1+1/(2k)$ for every $k \geq 4$.

To gain better insights on lower bounds, dense instances of many optimization problems has been intensively studied [2,17,18,16]. The Vertex Cover problem has been investigated in the case of dense graphs, where the number of edges is within a constant factor of n^2, by Karpinski and Zelikovsky [18], Eremeev [11], Clementi and Trevisan [8], later by Imamura and Iwama [14] as well as Cardinal et al. [6].

A k-uniform hypergraph is called dense if the number of edges is within a constant factor of n^k. Bar-Yehuda and Kehat [5] proved that there exists a polynomial time approximation algorithm for the Vertex Cover problem restricted to dense k-uniform hypergraphs with an approximation ratio better than k. After that, Cardinal et al. [7] gave a randomized approximation algorithm for the Vertex Cover problem in k-uniform hypergraphs with an approximation ratio parametrized by the density and the maximum degree of the underlying hypergraph. It yields for a larger class of k-uniform hypergraphs an approximation ratio better than k compared to the class of dense k-uniform hypergraphs.

A modified version of the approximation algorithm for the Vertex Cover problem in dense k-uniform hypergraphs in [7] can also be applied to the Vertex Cover problem restricted to dense k-partite k-uniform hypergraphs yielding an approximation ratio better than $k/2$ if the given hypergraph is dense enough.

In this paper, we design an improved approximation algorithm for the Vertex Cover problem restricted to dense k-partite k-uniform hypergraphs with an approximation ratio less than $k/2$ and prove that the achieved approximation ratio is almost tight assuming the Unique Games Conjecture.

2. Definitions and notations

Given a natural number $i \in \mathbb{N}$, we introduce for notational simplicity the set $[i]=\{1,..,i\}$ and set $[0]=\emptyset$. Let S be a finite set with cardinality s and $k \in [s]$. We will use the abbreviation $\binom{S}{k}=\{S' \subseteq S \mid |S'|=k\}$.

A *k-uniform hypergraph* $H=(V(H),E(H))$ consists of a set of vertices $V(H)$ and a collection $E(H) \subseteq \binom{V}{k}$ of edges. For a k-uniform hypergraph H and a vertex $v \in V(H)$, we define the *neighborhood* $N_H(v)$ of v by $\bigcup_{e \in \{e \in E \mid v \in e\}} e \setminus \{v\}$ and the *degree* $d_H(v)$ of v to be $|\{e \in E \mid v \in e\}|$. We extend this notion to subsets of $V(H)$, where $S \subseteq V(H)$ obtains the degree $d_H(S)$ by $|\{e \in E \mid S \subseteq e\}|$.

A *k-partite* k-uniform hypergraph $H=(V_1,..,V_k,E(H))$ is a k-uniform hypergraph such that V is a disjoint union of $V_1,\dots,V_k$ with $|V_i \cap e|=1$ for every $e \in E$ and $i \in [k]$. In the remainder, we assume that $|V_i| \geq |V_{i+1}|$ for all $i \in [k-1]$.

A *balanced* k-partite k-uniform hypergraph $H=(V_1,..,V_k,E(H))$ is a k-partite k-uniform hypergraph with $k|V_i|=|V|$ for all $i \in [k]$. We set $n=|V|$ and $m=|E|$ as usual.

For a k-partite k-uniform hypergraph $H=(V_1,..,V_k,E(H))$ and $v \in V_k$, we introduce the *v-induced hypergraph* $H(v)$, where the edge set of $H(v)$ is defined by $\{e \setminus \{v\} \mid v \in e \in E(H)\}$ and the vertex set of $H(v)$ is partitioned into $V_i \cap N_H(v)$ with $i \in [k-1]$.

A *vertex cover* of a k-uniform hypergraph $H=(V(H),E(H))$ is a subset C of $V(H)$ with the property that $e \cap C \neq \emptyset$ holds for all $e \in E(H)$. The Vertex Cover problem consists of finding a vertex cover of minimum size in a given k-uniform hypergraph. The Vertex Cover problem in k-partite k-uniform hypergraphs is the restricted problem, where a k-partite k-uniform hypergraph and its vertex partition is given as a part of the input.

We define a k-uniform hypergraph $H=(V(k),E(H))$ as *ϵ-dense* for an $\epsilon \in (0,1)$ if the condition $|E(H)| \geq \epsilon\binom{n}{k}$ holds.

We define a k-partite k-uniform hypergraph $H=(V_1,..,V_k,E(H))$ as *ϵ-dense* for an $\epsilon \in (0,1)$ if the following condition holds:

$$\left|E(H)\right| \geq \epsilon \prod_{i \in [k]} |V_i|$$

For $\ell \in [k-1]$, we introduce the notion of ℓ-wise ϵ-dense k-partite k-uniform hypergraphs. This definition naturally generalizes the notion of weak and strong density in graphs [18]. A similar definition was used in the work [7] and [17] in the context of dense k-uniform hypergraphs.

Given a k-partite k-uniform hypergraph H, we define H to be ℓ-wise ϵ-dense if there exist an $I \in \binom{[k]}{\ell}$ and an $\epsilon \in (0,1)$ such that for all $S \in \{S \subseteq V(H) \mid |V_i \cap S|=1 \text{ for all } i \in I; |S|=|I|\}$ the following condition holds.

$$d_H(S) \geq \epsilon \prod_{i \in [k] \setminus I} |V_i|$$