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Approximation with a fixed number of solutions of some multiobjective maximization problems ☆

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A B S T R A C T

We investigate the problem of approximating the Pareto set of some multiobjective optimization problems with a given number of solutions. Our purpose is to exploit general properties that many well studied problems satisfy. We derive existence and constructive approximation results for the biobjective versions of Max Submodular Symmetric Function (and special cases), Max Bisection, and Max Matching and also for the k-objective versions of Max Coverage, Heaviest Subgraph, Max Coloring of interval graphs.

1. Introduction

In multiobjective combinatorial optimization a solution is evaluated considering several objective functions and a major challenge in this context is to generate the set of efficient solutions or the Pareto set (see [12] about multiobjective combinatorial optimization). However, it is usually difficult to identify the efficient set mainly due to the fact that the number of efficient solutions can be exponential in the size of the input and moreover the associated decision problem is NP-complete even if the underlying single-objective problem can be solved in polynomial time (e.g. shortest path [12]). To handle these two difficulties, researchers have been interested in developing approximation algorithms with an a priori provable guarantee such as polynomial time constant approximation algorithms. Considering that all objectives have to be maximized, and for a positive $\rho \leqslant 1$, a ρ-approximation of Pareto set is a set of solutions that includes, for each efficient solution, a solution that is at least at a factor ρ on all objective values. Intuitively, the larger the size of the approximation set, the more accurate it can be.

It has been pointed out by Papadimitriou and Yannakakis [27] that, under certain general assumptions, there always exists a $(1-\varepsilon)$-approximation, with any given accuracy $\varepsilon > 0$, whose size is polynomial both in the size of the instance and in $1/\varepsilon$ but exponential in the number of criteria. In this result, the accuracy $\varepsilon > 0$ is given explicitly and a general upper bound on the size of the approximation set is given. When the number of solutions in the approximation set is limited, not every level of accuracy is possible. So, once the number of solutions is fixed in the approximation set of a multiobjective

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problem, the following questions are raised: What is the accuracy for which an approximation is guaranteed to exist? Which accuracy can be obtained in polynomial time?

In this paper we are interested in establishing for some multiobjective maximization problems the best approximation ratio of the set of efficient solutions when the size of the approximation set is given explicitly. We give three approaches that deal with biobjective and k-objective problems that allow us to obtain approximations of the set of efficient solutions with one or several solutions. More precisely, in a first approach, we consider a general maximization problem and establish a sufficient condition that guarantees the construction of a constant approximation of the Pareto set with an explicitly given number of solutions. As a corollary, we can construct a $(1-\varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions. In a second approach, we establish a necessary and sufficient condition for the construction of a constant approximation of the Pareto set with one solution. In a third approach, we establish a sufficient condition for the construction of one solution with approximation guarantee for k-objective selection problems. In these three approaches, if the corresponding solutions can be found in polynomial time then the biobjective or k-objective selection problems admits polynomial time approximation with one solution.

Properties defined in these three approaches apply to several problems previously studied in single-objective approximation. Thus we derive polynomial time constant approximations with one solution for Biobjective Max Bisection, Biobjective Max Partition, Biobjective Max Cut, Biobjective Max Set Splitting, Biobjective Max Matching and k-objective Heaviest Subgraph, k-objective Max q Colorable Subgraph and k-objective Max Coverage, but also for k-objective versions of some particular cases of Max Coverage: Max q Vertex Cover and Max q Selection. Some instances show that the given biobjective approximation ratios are the best we can expect. In addition Biobjective Max Partition, Biobjective Max Cut, Biobjective Max Set Splitting admit a $(1-\varepsilon)$-approximation of the Pareto set with $O(\frac{1}{\varepsilon})$ solutions.

Several results exist in the literature on the approximation of multiobjective combinatorial optimization problems. One can mention the existence of fully polynomial time approximation schemes for biobjective shortest path [19,33,32], knapsack [13,8], minimum spanning tree [27], scheduling problems [6], randomized fully polynomial time approximation scheme for matching [27], and polynomial time constant approximation for max cut [4], a biobjective scheduling problem [31] and the traveling salesman problem [5,26]. Note that [4], [26] and [31] are approximations with a single solution.

This article is organized as follows. In Section 2, we introduce basic concepts about multiobjective optimization and approximation. Section 3 is devoted to an approach for approximating some biobjective problems with one or several solutions. Section 4 presents a necessary and sufficient condition for approximating some biobjective problems with one solution within a constant factor. In Section 5 we establish an approach for approximating some multiobjective selection problems. Conclusions are provided in a final section.

2. Preliminaries on multiobjective optimization and approximation

Consider an instance of a multiobjective optimization problem with k criteria or objectives where X denotes the finite set of feasible solutions. Each solution $x \in X$ is represented in the objective space by its corresponding objective vector $w(x) = (w_1(x), \ldots, w_k(x))$. We assume that each objective has to be maximized.

From these k objectives, the dominance relation defined on X is defined as follows: a feasible solution x dominates a feasible solution x' if and only if $w_i(x) \geqslant w_i(x')$ for $i = 1, \ldots, k$ with at least one strict inequality. A solution x is *efficient* if and only if there is no other feasible solution $x' \in X$ such that x' dominates x, and its corresponding objective vector is said to be *non-dominated*. Usually, we are interested in finding a solution corresponding to each non-dominated objective vector. The set of all such solutions is called Pareto set.

For any $0 < \rho \leqslant 1$, a solution x is called a ρ-approximation of a solution x' if $w_i(x) \geqslant \rho \cdot w_i(x')$ for $i = 1, \ldots, k$. A set of feasible solutions X' is called a ρ-approximation of the set of all efficient solutions if, for every feasible solution $x \in X$, X' contains a feasible solution x' that is a ρ-approximation of x. If such a set exists, we say that the multiobjective problem admits a ρ-approximate Pareto set with $|X'|$ solutions.

An algorithm that outputs a ρ-approximation of a set of efficient solutions in polynomial time in the size of the input is called a ρ-approximation algorithm. In this case we say that the multiobjective problem admits a polynomial time ρ-approximate Pareto set.

Consider in the following a single-objective maximization problem P defined on a ground set $\mathcal{U}$. Every element $e \in \mathcal{U}$ has a non-negative weight $w(e)$. The goal is to find a feasible solution (subset of $\mathcal{U}$) with maximum weight. The weight of a solution S must satisfy the following scaling hypothesis: if $opt(I)$ denotes the optimum value of I, then $opt(I') = t \cdot opt(I)$, where I' is the same instance as I except that $w'(e) = t \cdot w(e)$. For example, the hypothesis holds when the weight of S is defined as the sum of its elements' weights, or $\min_{e \in S} w(e)$, etc.

In the k-objective version, called k-objective P, $k \geqslant 2$, every element $e \in \mathcal{U}$ has k non-negative weights $w_1(e), w_2(e), \ldots, w_k(e)$ and the goal is to find a Pareto set within the set of feasible solutions. Given an instance I of k-objective P, we denote by $opt_i(I)$ (or simply opt_i) the optimum value of I restricted to objective i, $i = 1, \ldots, k$. Here, the objective function on objective 1 is not necessarily of the same kind as on objective 2, but both satisfy the scaling hypothesis. For example, one objective can be additive (sum of element's weight) and the other can be bottleneck (min or max of element's weights).