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Near optimal algorithm for the shortest descending path on the surface of a convex terrain

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ABSTRACT

We study the problem of finding a shortest descending path (SDP) between a pair of points, called source (s) and destination (t), on the surface of a triangulated convex terrain with n faces. A path from s to t on a polyhedral terrain is *descending* if the height of a point p never increases while we move p along the path from s to t. Time and space complexity requirement of our algorithm are $O(\mu(n)\log n)$ and $O(\tau(n))$, respectively. Here $\mu(n)$ and $\tau(n)$ are time and space complexity requirement for finding shortest geodesic path (SGP) between a pair of points on the surface of a convex polyhedra. The best known bounds on $\mu(n)$ and $\tau(n)$ are both $O(n\log n)$ due to Schreiber and Sharir (2008) [11]. Earlier best known time and space complexity results of SDP on convex terrain were $O(n^2\log n)$ and $O(n^2)$, respectively, and appears in Roy et al. (2007) [10]. Thus our algorithm improves both time and space complexity requirement of SDP problem by almost a linear factor over earlier best known results.

1. Introduction

The SGP problem on the surface of polyhedron is studied extensively in the literature. Sharir and Schorr [12] presented an $O(n^3\log n)$ time algorithm for finding the geodesic shortest path between two points on the surface of a convex polyhedron with n vertices. Mitchell et al. [8] studied the generalized version of this problem where the restriction of convexity is removed. The time complexity of the proposed algorithm is $O(n^2\log n)$. A decade later, Chen and Han [6] improved the time complexity to $O(n^2)$. The best known algorithm for producing the SGP on the surface of a convex polyhedra is due to Schreiber and Sharir [11]. Time and space complexity requirement of their algorithm are both $O(n\log n)$.

De Berg and van Kreveld [7] first studied the problem of finding *descending paths* in the surface of a triangulated terrain $\mathcal{T}'$ with n faces. They preprocess $\mathcal{T}'$ in $O(n\log n)$ time to answer the following decision version of the problem in $O(\log n)$ time: Does there exist a descending path between two query points? Finding an SDP is a long-standing open problem [7] in the sense that no bound on the combinatorial or Euclidean length of the SDP between a pair of points on the surface of a polyhedral terrain is available in the literature. More than a decade later, Ahmed et al. [4] devised two FPTAS for a general terrain, both produced a $(1+\epsilon)$-factor approximation and are based on the idea of discretizing the terrain by adding Steiner points. The running time of their algorithms are $O(n^3\log\frac{X}{\epsilon})$, where X depends on some geometric parameters and n. Ahmed and Lubiw [1] explored more characteristics of an SDP to show that finding an exact SDP in a general terrain seems to be computationally hard.

Due to the inherent difficulty of finding SDP on the surface of general terrain, a lot of attention has been focused on devising algorithms for some interesting constrained cases. Roy et al. [10] studied the problem of computing an SDP on

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the surfaces of a convex terrain. Time and space complexity requirement of their algorithm are $O(n^2 \log n)$ and $O(n^2)$, respectively. They also provide an $O(n \log n)$ time optimal algorithm for restricted terrain when the boundaries of the given edge sequence are parallel. Roy et al. raised the question of finding an optimal solution even when we are additionally given a terrain edge sequence. Subsequently, Ahmed and Lubiw [2] gave a $(1+\epsilon)$-factor approximation algorithm for a given terrain edge sequence. They proved that the length of SDP in a given edge sequence is a convex function and used cone programming to provide an approximate solution. The running time of their algorithm is $O(n^{3.5} \log \frac{L}{\epsilon})$, where L is the length of the longest edge of the terrain. So, the optimal solution even for only a given terrain edge sequence is still unsolved. The results from Ahmed and Lubiw [1] have also been extended by the same authors in [3] to devise algorithms for generalizations of convex terrains and orthogonal terrains. Mount [9] demonstrated that SGPs on the surface of a convex polyhedron can be grouped into $\Theta(n^4)$ equivalence classes. In a very recent paper Ahmed et al. [5] showed that SDPs on the surface of a convex terrain can also be grouped into $\Theta(n^4)$ equivalence classes. They conjectured that SDPs on the surface of a convex polyhedron too will follow the same bound.

2. Our contribution

We provide an algorithm for finding SDP between a pair of points, called source (s) and destination (t), on the surface of a triangulated convex terrain with n faces. Time and space complexity requirement of our algorithm are $O(\mu(n) \log n)$ and $O(\tau(n))$, respectively. Here $\mu(n)$ and $\tau(n)$ are time and space complexity requirement for finding shortest geodesic path (SGP) between a pair of points on the surface of a convex polyhedra. The best known bound on $\mu(n)$ and $\tau(n)$ are both $O(n \log n)$ due to Schreiber and Sharir [11]. Thus time and space complexity requirement of our algorithm are $O(n \log^2 n)$ and $O(n \log n)$, respectively. Earlier best known time and space complexity results of SDP on convex terrain are $O(n^2 \log n)$ and $O(n^2)$, respectively, and appear in Roy et al. [10]. Thus our algorithm provides almost a linear factor improvement on time and space complexity results for SDP problem over earlier best known results. The main ingredients of our results are some very interesting characteristics of SDP on the surface convex terrain. A clever blending of these new characteristics of SDP along with the old characteristics of SDP and SGP in [5,8,10–12] will help us to devise an almost near optimal solution.

3. Preliminaries and notations

A terrain $\mathcal{T}'$ is a polyhedral 2D surface in 3D with the property that the vertical line at any point on the xy-plane intersects the surface of $\mathcal{T}'$ at most once. Thus, the projections of all the faces of a terrain on the xy-plane are pairwise disjoint in their interior. Each vertex p on the surface of the terrain is specified by a triple $(x(p), y(p), z(p))$. More formally, a terrain $\mathcal{T}'$ is the image of a real bivariate function ζ defined on a compact and connected domain $\mathcal{W}$ in the Euclidean plane, i.e., $\mathcal{T}' = \{(x, y, \zeta(x, y)), (x, y) \in \mathcal{W}\}$. Without loss of generality, we assume that all the faces of the terrain are triangles. The z-coordinate, $z(p)$, of a point $p \in \mathcal{T}'$ is also called the altitude of the point p.

A convex (concave) terrain $\mathcal{T}$ is the surface of a convex (concave) polyhedra in 3D with the property that the vertical line at any point on the xy-plane intersects the surface of $\mathcal{T}$ at most once. Let $\pi_d(a,b)$ and $\delta(a,b)$ denote the shortest descending path on the surface of $\mathcal{T}$ between a pair of points a and b and its length, respectively. So, SDP and $\pi_d(a,b)$ will be used in same sense as the context suits. A path $\pi(a,b)$ from a point a to a point b on the surface of the terrain is said to be a geodesic path if it entirely lies on the surface of the terrain, it is locally optimal (i.e., the length of the path cannot be reduced by small perturbation), it is not self-intersecting, and its intersection with a face is either empty or a straight line segment. The geodesic distance $dist(p,q)$ between a pair of points p and q on $\pi(a,b)$ is the length of the path from p to q along $\pi(a,b)$. A path $\pi_{geo}(a,b)$ is said to be the geodesic shortest path if the distance between a and b along $\pi_{geo}(a,b)$ is minimum among all possible geodesic paths from a point a to b. So, SGP and $\pi_{geo}(a,b)$ will be used in same sense as the context suits. Let $\delta_{geo}(a,b)$ denote the length of the path $\pi_{geo}(a,b)$. Let $E_{geo}(a,b)$ denote the sequence of terrain edges that the path $\pi_{geo}(a,b)$ traverses.

Definition 3.1. (See [8].) Let f and f' be a pair of faces in $\mathcal{T}$ that share an edge e. The *planar unfolding* of face f' with respect to face f is the image of the points of f' when rotated about the line e onto the plane containing f such that the points of f and the points of f' lie in two different sides of the edge e (i.e., faces f' and f do not overlap after unfolding).

Lemma 3.1. *(See [8].) For a pair of points s and t, if $\pi_{geo}(s,t)$ passes through an edge-sequence $E_{geo}(s,t)$ of a terrain, then in the planar unfolding $U(E_{geo}(s,t))$, $\pi_{geo}(s,t)$ is a straight line segment $[s^*, t^*]$, where s^* and t^* are the projections of s and t on $U(E_{geo}(s,t))$.*

4. Properties of the SDP on a convex terrain

In this section, we discuss important properties of SDP on the surface of $\mathcal{T}$.

Definition 4.1. A path $\pi(s,t)$ $(z(s) \geqslant z(t))$ on the surface of a terrain is a *descending path* if for every pair of points $p, q \in \pi(s,t)$, $dist(s,p) < dist(s,q)$ implies $z(p) \geqslant z(q)$.