Journal of Discrete Algorithms 9 (2011) 299–306

Contents lists available at ScienceDirect

Journal of Discrete Algorithms

www.elsevier.com/locate/jda

Contracting planar graphs to contractions of triangulations ☆

Marcin Kamiński [a], Daniël Paulusma [b], Dimitrios M. Thilikos [c,*]

[a] Département d'Informatique, Université Libre de Bruxelles, Boulevard du Triomphe CP212, B-1050 Brussels, Belgium
[b] School of Engineering and Computing Sciences, Science Laboratories, South Road, Durham DH1 3LE, England, United Kingdom
[c] Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR15784 Athens, Greece

ARTICLE INFO

Article history:
Received 12 December 2010
Received in revised form 24 February 2011
Accepted 7 March 2011
Available online 15 March 2011

Keywords:
Planar graph
Dual graph
Contraction
Topological minor
Fixed parameter tractable

ABSTRACT

For every graph H, there exists a polynomial-time algorithm deciding if a planar input graph G can be contracted to H. However, the degree of the polynomial depends on the size of H. We identify a class of graphs $\mathcal{C}$ such that for every fixed $H \in \mathcal{C}$, there exists a linear-time algorithm deciding whether a given planar graph G can be contracted to H. The class $\mathcal{C}$ is the closure of planar triangulated graphs under taking of contractions. In fact, we prove that a graph $H \in \mathcal{C}$ if and only if there exists a constant c_H such that if the treewidth of a graph is at least c_H, it contains H as a contraction. We also provide a characterization of $\mathcal{C}$ in terms of minimal forbidden contractions.

1. Introduction

We consider simple graphs without loops and multiple edges. For a graph G, let $V(G)$ be its vertex set and $E(G)$ its edge set. For notions not defined here, we refer the reader to the monograph on graph theory by Diestel [6].

1.1. Planar graphs

All graphs in this paper are planar. Plane graphs are always assumed to be drawn on the unit sphere and their edges are arbitrary polygonal arcs (not necessarily straight line segments).

Embeddings. In this work, we only need to distinguish between essentially different embeddings of a planar graph. This motivates the following definition. Two plane graphs G and H are *combinatorially equivalent* ($G \simeq H$) if there exists a homeomorphism of the unit sphere (in which they are embedded) that transforms one into the other. The relation of being combinatorially equivalent is reflexive, symmetric and transitive, and thus an equivalence relation. Let $\mathcal{G}$ be the class of all plane graphs isomorphic to a planar graph G, and let us consider the quotient set $\mathcal{G}/\simeq$. The equivalence classes (i.e., the elements of the quotient set) can be thought of as *embeddings*. In fact, we will work with embeddings but for simplicity, we will pick a plane graph representative for each embedding.

☆ This research was done while the last two authors were visiting Dèpartement d'Informatique of Université Libre de Bruxelles in January 2010. The authors thank Samuel Fiorini for his kind support. The work is also supported by Actions de Recherche Concertées (ARC) fund of Communauté française de Belgique, by the project "Kapodistrias" (AΠ 02839/28.07.2008) of the National and Kapodistrian University of Athens and by EPSRC Grant EP/G043434/1.

* Corresponding author.
E-mail addresses: marcin.kaminski@ulb.ac.be (M. Kamiński), daniel.paulusma@durham.ac.uk (D. Paulusma), sedthilk@math.uoa.gr (D.M. Thilikos).

1570-8667/$ – see front matter
doi:10.1016/j.jda.2011.03.010

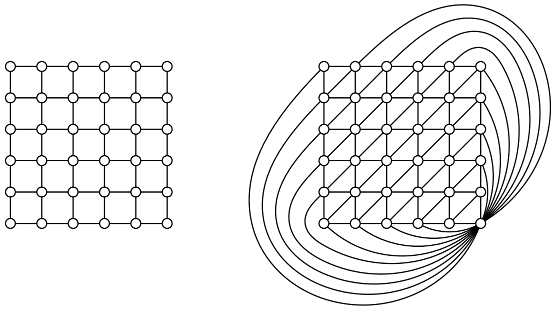

Fig. 1. Graphs M_6 and Γ_6, respectively.

Dual. The dual of a plane graph G will be denoted by G^*. Note that there is a one-to-one correspondence between the edges of G and the edges of G^*. We keep the convention that e^* is the edge of G^* corresponding to edge e of G.

Triangulation. A planar graph is called *triangulated* if it has an embedding in which every face is incident with exactly three vertices. Let us recall two useful facts related to planar 3-connected graphs that we will need later.

Lemma 1.1. *Triangulated planar graphs are* 3*-connected.*

Lemma 1.2. *A* 3*-connected planar graph has a unique embedding.*

For proofs of these lemmas, see for instance the monograph of Mohar and Thomassen [16], Lemma 2.3.3, p. 31 and Lemma 2.5.1, p. 39, respectively. From these two lemmas, every triangulated graph has a unique embedding.

Grids and walls. The $k \times k$ *grid* M_k has as its vertex set all pairs (i, j) for $i, j = 0, 1, \ldots, k-1$, and two vertices (i, j) and (i', j') are joined by an edge if and only if $|i - i'| + |j - j'| = 1$.

For $k \geqslant 2$, let Γ_k denote the graph obtained from M_k by triangulating its faces as follows: add an edge between vertices (i, j) and (i', j') if $i - i' = 1$ and $j' - j = 1$, and add an edge between corner vertex $(k-1, k-1)$ and every external vertex that is not already adjacent to $(k-1, k-1)$, i.e., every vertex (i, j) with $i \in \{0, k-1\}$ or $j \in \{0, k-1\}$, apart from the vertices $(k-2, k-1)$ and $(k-1, k-2)$. The graph Γ_k is called a *triangulated grid*. See Fig. 1 for graphs M_6 and Γ_6. The dual Γ_k^* of a triangulated grid is called a *wall*.

Treewidth and MSOL. The fragment of second-order logic where quantified relation symbols must have arity 1 is called *monadic second-order logic* (MSOL). A seminal result of Courcelle [4] is that on any class of graphs of bounded treewidth, every problem definable in monadic second-order logic can be solved in time linear in the number of vertices of the graph. Moreover, Courcelle's result holds not just when graphs are given in terms of their edge relation, but also when the domain of a structure encoding a graph G consists of the disjoint union of the set of vertices and the set of edges, as well as unary relations V and E to distinguish the vertices and the edges, respectively, and also a binary incidence relation I which denotes when a particular vertex is incident with a particular edge (thus, $I \subseteq V \times E$). The reader is referred to Courcelle [4] for more details and also for the definition of treewidth.

Lemma 1.3. *(See [4].) For every fixed k and every problem $\mathcal{P}$ expressible in MSOL, there exists a linear-time algorithm for $\mathcal{P}$ in the class of graphs of treewidth at most k.*

Below we recall some other results related to treewidth that we will need later in the paper.

Lemma 1.4. *(See (6.2) in [18].) Let $m \geqslant 1$ be an integer. Every planar graph with no $m \times m$ grid minor has treewidth $\leqslant 6m - 5$.*

Lemma 1.5. *(See (1.5) in [18].) If H is a planar graph with $|V(H)| + 2|E(H)| \leqslant n$, then H is isomorphic to a minor of the $2n \times 2n$ grid.*

Lemma 1.6. *(See Theorem 6 in [2].) For any plane graph G and its dual G^*, $\mathbf{tw}(G)^* \leqslant \mathbf{tw}(G) + 1$.*

Lemma 1.7. *(See Theorem 1.1 in [1].) For every fixed k, there exists a linear-time algorithm deciding whether the input graph has treewidth at most k.*

Pasting along vertices and edges. Let G be a graph with induced subgraphs G_1 and G_2 such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Let $G_1 \cap G_2$ denote the subgraph of G induced by $V(G_1) \cap V(G_2)$. Then we say that G arises