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Computing sharp 2-factors in claw-free graphs ☆,☆☆

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ABSTRACT

In a previous paper we obtained an upper bound for the minimum number of components of a 2-factor in a claw-free graph. This bound is sharp in the sense that there exist infinitely many claw-free graphs for which the bound is tight. In this paper we extend these results by presenting a polynomial algorithm that constructs a 2-factor of a claw-free graph with minimum degree at least four whose number of components meets this bound. As a byproduct we show that the problem of obtaining a minimum 2-factor (if it exists) is polynomially solvable for a subclass of claw-free graphs. As another byproduct we give a short constructive proof for a result of Ryjáček, Saito and Schelp.

1. Introduction

In this paper we consider 2-factors of claw-free graphs. Graph factors are well-studied. See [16] for a survey. Our motivation to study 2-factors goes back to the well-known NP-complete decision problem H-Cycle (cf. [9]) in which the problem is to decide whether a given graph has a hamiltonian cycle, i.e., a connected 2-regular spanning subgraph. In the related problem 2-Factor the connectivity condition is dropped, hence the problem is to decide whether a given graph admits a 2-factor, i.e., a 2-regular spanning subgraph. This makes the problem considerably easier in the algorithmic sense: it is well known that 2-Factor can be solved in polynomial time by matching techniques, and a 2-factor can be constructed in polynomial time if the answer is YES (cf. [14]). Clearly, a hamiltonian cycle is a 2-factor consisting of one component, and the minimum number of components of a 2-factor can be seen as a measure for how far a graph is from being hamiltonian. So, from an algorithmic viewpoint a natural question is to consider the problem of determining a 2-factor of a given graph with a minimum number of components. Obviously, this is an NP-hard problem. Hence it makes sense to search for 2-factors with a reasonably small number of components if we aim for polynomial time algorithms. For this research we have restricted ourselves to the class of claw-free graphs. This is a rich class containing, e.g., the class of line graphs and the class of complements of triangle-free graphs. It is also a very well-studied graph class, both within structural graph theory and within algorithmic graph theory; see [7] for a survey. Furthermore, computing a 2-factor with a minimum number of components remains NP-hard for the class of claw-free graphs.

In [1] we already obtained an upper bound for the minimum number of components of a 2-factor in a claw-free graph. This bound is sharp in the sense that there exist infinitely many claw-free graphs for which the bound is tight; we will specify this later. When considering the related complexity problems, we soon realized that the proof methods used in [1] need to be extended in order to obtain a polynomial algorithm that constructs a corresponding 2-factor, e.g., a 2-factor whose number of components is at most our upper bound. In the present paper we present this polynomial time algorithm.

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2. Terminology and background

We consider graphs that are finite, undirected and simple, i.e., without multiple edges and loops. For notation and terminology not defined in this paper we refer to [4].

Let $G = (V(G), E(G))$ be a graph of order $|G| = |V(G)| = n$ and of size $e(G) = |E(G)|$. The neighbor set of a vertex x in G is denoted by $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$, and its cardinality by $d_G(x)$. We denote the minimum (vertex) degree of G by $\delta_G = \min\{d_G(x) \mid x \in V(G)\}$. If no confusion can arise we often omit the subscripts.

Let K_n denote the complete graph on n vertices. A graph F is called a 2-*factor* of a graph G if F is a 2-regular spanning subgraph of G, i.e., if F is a subgraph of G with $V(F) = V(G)$ and $d_F(x) = 2$ for all $x \in V(F)$. A *claw-free* graph is a graph that does not contain an induced subgraph isomorphic to the four-vertex star $K_{1,3} = (\{u, a, b, c\}, \{ua, ub, uc\})$.

2.1. Known results

Several interesting problems are still open for claw-free graphs such as the conjecture of Matthews and Sumner [15] that every 4-connected claw-free graph is hamiltonian. However, there is quite a lot known on 2-factors in claw-free graphs, including some very recent results. Results of both Choudum and Paulraj [3] and Egawa and Ota [5] imply that every claw-free graph with $\delta \geqslant 4$ contains a 2-factor.

Theorem 1. *(See [3,5].) A claw-free graph with $\delta \geqslant 4$ has a 2-factor.*

We observe that every 4-connected claw-free graph has minimum degree at least four, and hence has a 2-factor. A 2-connected claw-free graph already has a 2-factor if $\delta = 3$ [20]. However, in general a claw-free graph with $\delta \leqslant 3$ does not have to contain a 2-factor. Examples are easily obtained.

Faudree et al. [6] showed that every claw-free graph with $\delta \geqslant 4$ has a 2-factor with at most $6n/(\delta + 2) - 1$ components. Gould and Jacobson [11] proved that, for every integer $k \geqslant 2$, every claw-free graph of order $n \geqslant 16k^3$ with $\delta \geqslant n/k$ has a 2-factor with at most k components. Fronček, Ryjáček and Skupień [8] showed that, for every integer $k \geqslant 4$, every claw-free graph G of order $n \geqslant 3k^2 - 3$ with $\delta \geqslant 3k - 4$ and $\sigma_k > n + k^2 - 4k + 7$ has a 2-factor with at most $k - 1$ components. Here σ_k denotes the minimum degree sum of any k mutually nonadjacent vertices.

If a graph G is claw-free, 2-connected and has $\delta \geqslant 4$, then G has a 2-factor with at most $(n + 1)/4$ components [13]. If a graph G is claw-free, 3-connected and has $\delta \geqslant 4$, then G has a 2-factor with at most $2n/15$ components [13].

In [1] we considered claw-free graphs with $\delta \geqslant 4$. Our motivation for this is as follows. We first note that the number of components of a 2-factor in any graph on n vertices is obviously at most $n/3$. For claw-free graphs with $\delta = 2$ that have a 2-factor we cannot do better than this trivial upper bound. This is clear from considering a disjoint set of triangles (cycles on three vertices). For claw-free graphs with $\delta = 3$ that have a 2-factor, the upper bound $n/3$ on its number of components is also tight, as shown in [1]. Hence, in order to get a nontrivial result it is natural to consider claw-free graphs with $\delta \geqslant 4$.

Our two main results in [1] provide answers to two open questions posed in [20].

Theorem 2. *(See [1].) A claw-free graph G on n vertices with $\delta \geqslant 5$ has a 2-factor with at most $(n - 3)/(\delta - 1)$ components unless G is isomorphic to K_n.*

Theorem 3. *(See [1].) A claw-free graph G on n vertices with $\delta = 4$ has a 2-factor with at most $(5n - 14)/18$ components, unless G belongs to a finite class of exceptional graphs.*

Both results are tight in the following sense. Let $f_2(G)$ denote the minimum number of components in a 2-factor of G. Then in [20], for every integer $d \geqslant 4$, an infinite family $\{F_i^d\}$ of claw-free graphs with $\delta(F_i^d) \geqslant d$ is given such that $f_2(F_i^d) > |F_i^d|/d \geqslant |F_i^d|/\delta(F_i^d)$. This shows we cannot replace $\delta - 1$ by δ in Theorem 2. The bound in Theorem 3 is tight in the following sense. There exists an infinite family $\{H_i\}$ of claw-free graphs with $\delta(H_i) = 4$ such that

$$\lim_{|H_i| \to \infty} \frac{f_2(H_i)}{|H_i|} = \frac{5}{18}.$$

This family can be found in [20] as well.

The exceptional graphs of Theorem 3 have at most seventeen vertices. They are described in [1], and we will not specify them here. In [1] we also explain that Theorems 2 and 3 together improve the previously mentioned result of Faudree et al. [6] and that Theorem 2 also improves the previously mentioned result of Gould and Jacobson [11].

2.2. Results of this paper

The proofs in [1] do not yield algorithms for constructing 2-factors that satisfy the upper bounds in Theorems 2 and 3. In the remainder of this paper we will develop a new approach to these problems in order to establish polynomial algorithms that construct 2-factors of claw-free graphs with minimum degree at least four. Using our results in [1] we show that