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On minimum metric dimension of honeycomb networks

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Abstract

A minimum metric basis is a minimum set W of vertices of a graph $G(V, E)$ such that for every pair of vertices u and v of G, there exists a vertex $w \in W$ with the condition that the length of a shortest path from u to w is different from the length of a shortest path from v to w. The honeycomb and hexagonal networks are popular mesh-derived parallel architectures. Using the duality of these networks we determine minimum metric bases for hexagonal and honeycomb networks.

Keywords: Metric basis; Metric dimension; Hexagonal network; Honeycomb network; Dual of a graph

1. Introduction and background

Multiprocessor interconnection networks are often required to connect thousands of homogeneously replicated processor-memory pairs, each of which is called a *processing node*. Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Design and use of multiprocessor interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful microprocessors and memory chips [2].

It is known that there exist three regular plane tessellations, composed of the same kind of regular polygons: triangular, square, and hexagonal. They are the basis for the design of direct interconnection networks with highly competitive overall performance. Grid connected computers and tori (Figs. 1(a) and (b)) are based on regular square tessellations, and are popular and well-known models for parallel processing.

Built recursively using the hexagon tessellation [13], *honeycomb networks* (Fig. 1(c)) are widely used in computer graphics [7], cellular phone base stations [9], image processing, and in chemistry as the representation of benzenoid hydrocarbons. Honeycomb networks are better in terms of degree, diameter, total number of links, cost and the bisection width than mesh connected planar graphs. Stojmenovic [13] has studied the topological properties of honeycomb networks, routing in honeycomb networks and honeycomb torus networks. Parhami [10] gave a unified formulation

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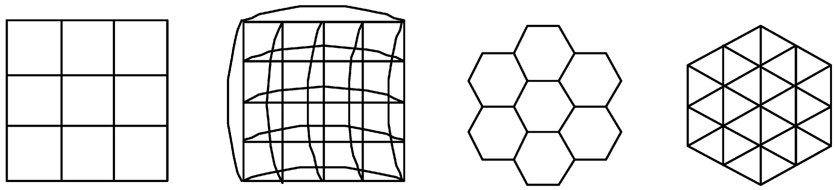

(a) 4 x 4 Grid network (b) Torus network (c) Honeycomb network (d) Hexagonal network

Fig. 1. Graphs obtained by regular plane tessellations.

for the honeycomb and the diamond networks. The embedding of honeycomb networks into trees, hypercubes and other networks was suggested as an open problem in [13].

The triangular tessellation is used to define *Hexagonal network* (Fig. 1(d)) and this is widely studied in [2]. An addressing scheme for hexagonal networks, and its corresponding routing and broadcasting algorithms were proposed by Chen et al. [2].

2. An overview of the paper

A *metric basis* for a graph $G(V, E)$ is a set $W \subseteq V$ such that for each pair of vertices u and v of $V \setminus W$, there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. A *minimum metric basis* is a metric basis of minimum cardinality. The cardinality of a minimum metric basis of G is called *minimum metric dimension* and is denoted by $md(G)$; the members of a minimum metric basis are called *landmarks*. The *minimum metric dimension* (MMD) *problem* is to find a minimum metric basis.

The problem of finding the metric dimension of a graph was first studied by Harary and Melter [5]. Melter and Tomescu [8] studied the metric dimension problem for grid graphs. Khuller et al. [6] describe the application of this problem in the field of computer science and robotics. This problem has been studied for trees, multi-dimensional grids [6], Petersen graphs [1], and Torus Networks [11]. Surprisingly, there is not much relevant work in the literature. The algorithmic complexity status of MMD problem is not known to even simple graphs such as co-graphs, interval graphs, Cayley graphs etc.

It is interesting to learn [6] that a graph has metric dimension 1 if and only if it is a path. The problem of computing the metric dimension of trees is solved in linear time [6]. If G has p vertices then it is clear that $1 \leqslant md(G) \leqslant p - 1$. Also $md(K_p) = p - 1$, $md(C_p) = 2$, and $md(K_{m,n}) = m + n - 2$, where K_p, C_p, and $K_{m,n}$ are the complete graph, the cycle, and the complete bipartite graph respectively [5]. Garey and Johnson [3] proved that this problem is *NP*-complete for general graphs by a reduction from 3-dimensional matching.

The aim of this paper is to find the minimum metric dimension of the honeycomb networks. Hexagonal network $HX(n)$ has a simple distance property like the two-dimensional grid. We first locate a minimum metric basis of $HX(n)$. By making use of the fact that honeycomb networks $HC(n)$ and hexagonal networks $HX(n)$ are dual networks, we derive a minimum metric basis for $HC(n)$. We prove that the minimum metric dimension of the honeycomb networks of size n is 3.

3. Properties of honeycomb networks

Honeycomb networks can be built from hexagons in various ways. The honeycomb network $HC(1)$ is a hexagon. The honeycomb network $HC(2)$ is obtained by adding six hexagons to the boundary edges of $HC(1)$. Inductively, honeycomb network $HC(n)$ is obtained from $HC(n - 1)$ by adding a layer of hexagons around the boundary of $HC(n - 1)$. For instance, Fig. 1(c) is $HC(2)$. The parameter n of $HC(n)$ is determined as the number of hexagons between the centre and boundary of $HC(n)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2 - 3n$ respectively. The diameter is $4n - 1$ [13].

In order to view the honeycomb $HC(n)$ as a dual of the hexagonal network $HX(n)$, let us recall the definition of a dual graph. Let G be a planar graph. The dual of G, denoted by $G^\star$, is a graph whose vertex set is the set of faces of G, where two vertices $f^\star$ and $g^\star$ in $G^\star$ are joined by an edge $e^\star$ if the faces f and g are separated by the edge e. Clearly the number of vertices of $G^\star$ is equal to the number of faces of G and the number of edges of $G^\star$ is equal to