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Small k-pyramids and the complexity of determining k

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ABSTRACT

Motivated by the computational complexity of determining whether a graph is hamiltonian, we study under algorithmic aspects a class of polyhedra called k-pyramids, introduced in [31], and discuss related applications. We prove that determining whether a given graph is the 1-skeleton of a k-pyramid, and if so whether it is belted or not, can be done in polynomial time for $k \leq 3$. The impact on hamiltonicity follows from the traceability of all 2-pyramids and non-belted 3-pyramids, and from the hamiltonicity of all non-belted 2-pyramids. The algorithm can also be used to determine the outcome for larger values of k, but the complexity increases exponentially with k. Lastly, we present applications of the algorithm, and improve the known bounds for the minimal cardinality of systems of bases called foundations in graph families with interesting properties concerning traceability and hamiltonicity.

1. Introduction

Determining whether a given graph is hamiltonian is a classical NP-complete problem [18]. Based on this, Garey, Johnson, and Tarjan [12] showed that determining traceability is an NP-complete problem, too. Even restricted to planar, cubic, 3-connected graphs, determining hamiltonicity remains NP-complete [12]. Thus, it is also NP-complete for the class of polyhedral graphs. In this contribution, we focus on k-pyramids, a class of polyhedra which generalizes those having Halin graphs as 1-skeleta and includes pyramids and prisms. Other generalizations of Halin graphs and investigations of their hamiltonian properties have already been made by Skowrońska [25], Skowrońska and Sysło [26], Skupień [27], and Malik et al. [22].

All graphs in this paper are finite, undirected, connected, and contain neither loops nor edges. Such a graph is called *polyhedral* if it is planar and 3-connected. (Recall that there is a bijection between polyhedral graphs and the 1-skeleta of polytopes by Steinitz's famous theorem [28].) In order to study a polyhedral graph G, we will need the concepts of dual graph and independent dominating set. On the one hand, constructing the dual graph is, algorithmically speaking, easily dealt with. On the other hand, it was shown by Garey and Johnson [11] that the problem of finding an independent dominating set of minimal cardinality (MIDS) is NP-complete. It remains NP-complete if restricted to line graphs [30], bipartite graphs [9,16] and dually chordal graphs [5]. Polynomial-time algorithms exist for many families of graphs, e.g. chordal graphs [10], interval and circular-arc graphs [7], cocomparability graphs [20], asteroidal triple-free graphs [6], and series–parallel graphs [24,14]. For more details, see the excellent article by Manlove [23]. In the same paper it was shown that MIDS, even restricted to cubic planar graphs, still is NP-complete.

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There exist several results concerning exact exponential time algorithms for MIDS. For arbitrary graphs with n vertices we have $O(1.3575^n)$, given in [13], and $O^*(3^{n/2})$ [21]. In the same paper they prove that for graphs with degree bounded by 3, we have $O^*(2^{0.465n})$.

Motivated by the computational complexity of determining hamiltonicity and traceability, and Problem 3 raised in [31], we present two algorithms with $O(n^4)$ time complexity that, given a connected finite graph without loops and multiple edges, (i) determine the minimal value k for which the graph is the 1-skeleton of a k-pyramid, and (ii) whether the graph is belted or not. In the remainder of this paper, we will address the following subsequent problems for a given polyhedral input graph G:

(P1) Is there a natural number k such that G is isomorphic to a k-pyramid?
(P2) Compute k^*, the minimal k for which (P1) holds.
(P3) Let (P2) be satisfied, and $k^* \in \{2, 3\}$. Is G non-belted?

2. Definitions

A polytope P in $\mathbb{R}^3$ is said to be *hamiltonian* (*traceable*) if its 1-skeleton – which is a polyhedral graph – has a hamiltonian cycle (path). Two facets of P will be called *neighbouring* if they share a common edge. A polytope or one of its facets is called *simple* if each of its vertices lies on precisely three edges of the polytope. The 1-skeleton of a simple polyhedron is a *cubic* graph, i.e. a graph in which all vertices are *cubic*, i.e. have degree 3. We call a face of a planar graph *cubic* if all of its boundary vertices are cubic.

P is called a *k-pyramid* [31] if there exist at most k pairwise disjoint simple facets $F_1, \ldots, F_k$, called *bases*, such that every other facet has some neighbouring base. For $k \geq 2$, we call P *belted* if some pair of bases has no common neighbouring facet. In particular, we say that a belted 2-pyramid with basic cycles C_1, C_2 (i.e., C_1 and C_2 are disjoint cycles that bound the bases) is *simply belted* if every vertex of the unique cycle disjoint from $C_1 \cup C_2$ in P is of degree 3. Denote the family of the 1-skeleta of (cubic, belted) k-pyramids by $\mathscr{P}_k$ ($\mathscr{P}_k^3$, $\overline{\mathscr{P}}_k$), and write $\bigcup_k \mathscr{P}_k = \mathscr{P}$. ($\mathscr{P}^3$ and $\overline{\mathscr{P}}$ are defined analogously.) Notice that $\mathscr{P}_1$ contains precisely the well-known Halin graphs. It was proven by Bondy [3] that they are all hamiltonian – even 1-hamiltonian, i.e. they are hamiltonian and remain hamiltonian when removing an arbitrary vertex.

Consider $G \in \mathscr{P}_k$. G is a graph with at most k pairwise disjoint cycles called *basic cycles* which bound the bases. For $G \in \mathscr{P}$, define $k^* = k^*(G) = \min\{k \in \mathbb{N} : G \in \mathscr{P}_k\}$. For $G \notin \mathscr{P}$, put $k^*(G) = \infty$. Call such a set of (k^*) k basic cycles a (*minimal*) *foundation*. Notice that two foundations of the same graph may have different cardinalities, as for the n-gonal prism, $n \geq 6$. Also, observe that minimal foundations can be unique (see Fig. 1), but need not be (see Figs. 3 and 4); two minimal foundations may even have no basic cycle in common, as is the case for the graph in Fig. 3, or the hexagonal prism. Keep in mind that G need not be cubic, but all vertices on the basic cycles must have degree 3, i.e. their corresponding facets must be simple.

The *dual graph* G^* of a planar graph G has vertices each of which corresponds to a face of G, and has faces each of which corresponds to a vertex of G. Two vertices in G^* are connected by an edge iff the corresponding faces in G are adjacent. We shall write $g(F) = v$ if F is a face of G, and v its corresponding vertex in G^*.

Now let G be a polyhedral graph. We will write $G = (V(G), E(G))$ with $|V(G)| = n$ and $|E(G)| = m$, and put $G^* = (V(G^*), E(G^*))$, where $V(G^*) = \{v_1, \ldots, v_f\}$. For a vertex $v \in V(G)$ we denote by $N(v)$ the set of all neighbours of v, and put $N[v] = N(v) \cup \{v\}$. Additionally, let $\mathscr{F}$ ($\mathscr{F}^3$) be the set of all (cubic) faces of G. We have $|\mathscr{F}| = f$ and put $|\mathscr{F}^3| = r$.

An *independent set* $X \subset V(G)$ of a graph G contains exclusively vertices that are pairwise non-adjacent. A set $Y \subseteq V(G)$ is called *dominating* if every vertex of G is either in Y, or a neighbour of a vertex in Y. We denote the minimal cardinality of an independent dominating set in G by $i(G)$. We recall that Berge [2] observed that an independent set is dominating iff it is a maximal independent set. Thus $i(G)$ is equal to the minimum cardinality of a maximal independent set in G. Berge also observed that every maximal independent set in a graph G is a minimal dominating set of G.

3. Computing k^* and testing beltedness

Let G be an arbitrary graph. For G to be polyhedral it must be planar and 3-connected. We present in the following two algorithms. The first (see Section 3.1) covers the case when the input graph is known to be cubic, whereas the second (see Section 3.2) deals with arbitrary input graphs. The purpose of these algorithms is not to showcase optimal run-times, but to demonstrate that the algorithmic problem has polynomial complexity. We output k^*, the minimal k for which $G \in \mathscr{P}_k$ (lower and upper bounds for k are provided in Appendix A). If $G \notin \mathscr{P}$, we output $k^* = \infty$.

Test 1 *(Planarity).* We test the planarity of G using the Boyer–Myrvold planarity test and embedding algorithm (BMEA) [4], which has a complexity of $O(n)$. Notice that after the embedding we can traverse the $f = 2 + m - n \leq 2n - 4$ faces of $\mathscr{F}$ in $O(n)$ time.

Test 2 *(3-Connectedness).* For any pair of vertices u, v in G we test whether $G \setminus \{u, v\}$ is connected. To verify whether a graph is connected or not can be done, for example, with a breadth-first search, which runs in $O(n + m)$ time. Accordingly, this processing step has a time complexity of $O(n^3)$ for planar graphs.