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On the strong chromatic index and maximum induced matching of tree-cographs, permutation graphs and chordal bipartite graphs

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ABSTRACT

We show that there exist linear-time algorithms that compute the strong chromatic index and a maximum induced matching of tree-cographs when the decomposition tree is a part of the input. We also show that there exist efficient algorithms for the strong chromatic index of (bipartite) permutation graphs and of chordal bipartite graphs.

1. Introduction

Definition 1. (See [7].) An *induced matching* in a graph G is a set of edges, no two of which meet a common vertex or are joined by an edge of G. The size of an induced matching is the number of edges in the induced matching. An induced matching is maximum if its size is largest among all possible induced matchings.

Definition 2. (See [17].) Let $G = (V, E)$ be a graph. A *strong edge coloring* of G is a proper edge coloring such that no edge is adjacent to two edges of the same color. A strong edge-coloring of a graph is a partition of its edges into induced matchings. The *strong chromatic index* of G is the minimal integer k such that G has a strong edge coloring with k colors. We denote the strong chromatic index of G by $s\chi'(G)$.

Equivalently, a strong edge coloring of G is a vertex coloring of $L(G)^2$, the square of the linegraph of G. The strong chromatic index problem can be solved in polynomial time for chordal graphs [7] and for partial k-trees [37], and can be solved in linear time for trees [15]. However, it is NP-complete to find the strong chromatic index for general graphs [7,34,39] or even for planar bipartite graphs [26]. In this paper, we show that there exist linear-time algorithms that compute the

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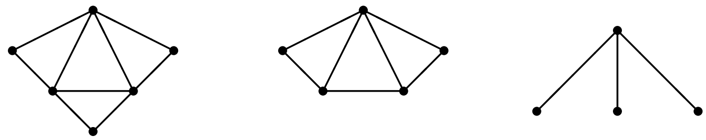

Fig. 1. A 3-sun, a gem and a claw.

strong chromatic index and a maximum induced matching of tree-cographs when the decomposition tree is a part of the input. We also show that there exist efficient algorithms for the strong chromatic index of (bipartite) permutation graphs and of chordal bipartite graphs.

The class of tree-cographs was introduced by Tinhofer in [40].

Definition 3. Tree-cographs are defined recursively by the following rules.

1. Every tree is a tree-cograph.
2. If G is a tree-cograph then also the complement $\bar{G}$ of G is a tree-cograph.
3. For $k \geq 2$, if $G_1, \dots, G_k$ are connected tree-cographs then also the disjoint union is a tree-cograph.

Let G be a tree-cograph. A decomposition tree for G consists of a rooted binary tree T in which each internal node, including the root, is labeled as a join node $\otimes$ or a union node $\oplus$. The leaves of T are labeled by trees or complements of trees. It is easy to see that a decomposition tree for a tree-cograph G can be obtained in $O(n^3)$ time.

2. The strong chromatic index of tree-cographs

The *linegraph* $L(G)$ of a graph G is the intersection graph of the edges of G [4]. It is well-known that, when G is a tree then the linegraph $L(G)$ of G is a claw-free blockgraph [23] (see Fig. 1 for an illustration of claw). A graph is *chordal* if it has no induced cycles of length more than three [13]. Notice that blockgraphs are chordal.

A vertex x in a graph G is *simplicial* if its neighborhood $N(x)$ induces a clique in G. Chordal graphs are characterized by the property of having a *perfect elimination ordering*, which is an ordering $[v_1, \dots, v_n]$ of the vertices of G such that v_i is simplicial in the graph induced by $\{v_i, \dots, v_n\}$. A perfect elimination ordering of a chordal graph can be computed in linear time [36]. This implies that chordal graphs have at most n maximal cliques, and the clique number can be computed in linear time, where the *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G.

Theorem 1. *(See [7].) If G is a chordal graph then $L(G)^2$ is also chordal.*

Theorem 2. *(See [9].) Let $k \in \mathbb{N}$ and let $k \geq 4$. Let G be a graph and assume that G has no induced cycles of length at least k. Then $L(G)^2$ has no induced cycles of length at least k.*

Lemma 1. *Tree-cographs have no induced cycles of length more than four.*

Proof. Let G be a tree-cograph. First observe that trees are bipartite. It follows that complements of trees have no induced cycles of length more than four.

We prove the claim by induction on the depth of a decomposition tree for G. If G is the union of two tree-cographs G_1 and G_2 then the claim follows by induction since any induced cycle is contained in one of G_1 and G_2. Assume G is the join of two tree-cographs G_1 and G_2. Assume that G has an induced cycle C of length at least five. We may assume that C has at least one vertex in each of G_1 and G_2. As one of G_1 and G_2 has more than two vertices of C, C has a vertex of degree at least three, which is a contradiction. □

Lemma 2. *Let T be a tree. Then $L(\bar{T})^2$ is a clique.*

Proof. Consider two non-edges $\{a, b\}$ and $\{p, q\}$ of T. If the non-edges share an endpoint then they are adjacent in $L(\bar{T})^2$ since they are already adjacent in $L(\bar{T})$. Otherwise, since T is a tree, at least one pair of $\{a, p\}$, $\{a, q\}$, $\{b, p\}$ and $\{b, q\}$ is a non-edge in T, otherwise T has a 4-cycle. By definition, $\{a, b\}$ and $\{p, q\}$ are adjacent in $L(\bar{T})^2$. □

If G is the union of two tree-cographs G_1 and G_2 then

$$\omega\big(L(G)^2\big) = \max\big\{\omega\big(L(G_1)^2\big), \omega\big(L(G_2)^2\big)\big\}.$$

The following lemma deals with the join of two tree-cographs.