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Computing upward topological book embeddings of upward planar digraphs ☆

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ABSTRACT

We describe a unified approach for studying book, point-set, and simultaneous embeddability problems of upward planar digraphs. The approach is based on a linear time strategy to compute an upward planar drawing of an upward planar digraph such that all vertices are collinear. Besides having impact in relevant application domains of graph drawing and computational geometry, the presented results open new research directions in the area of upward planarity with constraints of the positions of the vertices.

1. Introduction and overview

An *upward planar digraph* is a planar digraph that has a drawing in the plane without edge crossings and such that each directed edge appears as a directed curve that is monotonically increasing with respect to a common "upward" direction. For example, if the y-axis, directed upward, gives the common upward direction, then the points along the curve representing a directed edge u, v have y-coordinates that must increase as the curve is traversed from u to v.

Clearly, upward planar digraphs are necessarily acyclic. They have been studied extensively, in part motivated by the desire to produce good visualizations for process scheduling problems. For some application areas, such as knowledge engineering and project management, it is customary to align vertices on a common line l. It is then desirable to avoid the visual distraction created by edge crossings of l. Thus, one wants to draw the vertices on the y-axis, directed upward, say, with each directed edge drawn monotonically increasing, and lying in either the left or the right half-plane. In the language of book embeddings, described in more detail later, such a drawing might be called an *upward* two-page book embedding, with vertices lying on a vertical *spine* directed upward, and with edges drawn in either the left or right *page*, and drawn monotonically increasing with respect to the directed spine.

Unfortunately, not every upward planar digraph can be drawn in this way. For example, the underlying undirected planar graph might require more than two pages. Such is the case for the upward planar directed graph shown in Fig. 1.

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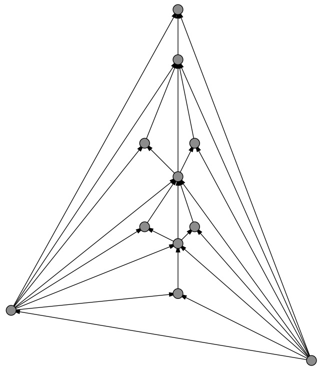

Fig. 1. An upward planar digraph whose page number is greater than two.

Its underlying undirected planar graph is the Goldner–Harary graph, which is not sub-hamiltonian and which therefore, according to the literature, has page number greater than two [5].

This motivates the following general question addressed in this paper:

Question. Is there a constant c such that every upward planar digraph G has a drawing such that all the vertices are positioned on the same directed line l, and such that each directed edge is monotonically increasing in the l direction, and crosses l at most c times? Is there an efficient algorithm to test whether such a drawing is possible, and if so, to produce a drawing?

First, we remark informally on how an upward planar digraph G is given as an input to a problem or an algorithm. Any drawing Γ of G, where Γ lies in the plane, is associated with a collection of cycles bounding the faces of Γ; one of these faces is the unbounded, external face. Drawings of G that have the same external face and the same combinatorial structure define an equivalence class. It is a description of such an equivalence class that is given as input to problems in this paper. Thus not only is the digraph G given, but also, the combinatorial structure of a proper drawing of G, i.e., one that exhibits the defining upward planarity properties of G. When we ask whether G can be drawn with certain additional properties, it is understood that the output drawing must preserve all the combinatorial structure specified in the input.

We now describe informally the main results of our paper and how they fit into the context of embedding problems for upward planar digraphs. We also compare results for embedding upward planar digraphs with the corresponding problems for planar digraphs, where there is no requirement to draw directed edges in an "upward" manner. The theorems referred to in this introduction require the technical preliminaries and terminology given in later sections to state fully and precisely.

2-page topological book embeddability (edges can cross the spine)	Planar graphs	Upward planar digraphs
Without prescribed ordering of the vertices on the spine	At most one spine crossing per edge is sufficient [18].	Minimization of the total number of spine-crossings is possible for outerplanar st-digraphs[a] [44]. **Our result:** At most one spine crossing per edge is sufficient (Theorem 1).
With prescribed ordering of the vertices on the spine	$3n+2$ spine crossings per edge are sufficient [3], and $\Omega(n)$ spine crossings per edge are sometimes necessary [3,47].	**Our result:** At most $n-4$ spine crossings per edge are sufficient and this is worst-case optimal (Theorem 2).

[a] An st-digraph is an acyclic digraph with exactly one source s and exactly one sink t.

Note that there are two scenarios for the general question posed above, depending on whether or not an ordering ρ of the vertices along the spine l is given. We prove in Theorem 1 that, when no such ρ is given, every upward planar digraph can be drawn with vertices on a directed spine l, such that no edge crosses l more than once. A drawing can be computed in linear time, based on any upward planar embedding. When a ρ is given, then we prove in Theorem 2 that a drawing with at most $n-4$ spine crossings per edge, which is worst-case optimal, can be computed in quadratic time, based on any upward planar embedding.

An additional benefit of our techniques for computing topological book embeddings is that they can be adapted to provide techniques for two other types of embeddings, namely, point-set embeddings and simultaneous embeddings. Indeed the connection we make here between topological book embeddings and these other types of embeddings may provide a useful insight for future work.