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# Universal point sets for planar three-trees ☆

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ABSTRACT

For every $n \in \mathbb{N}$, we present a set $S_n$ of $O(n^{3/2} \log n)$ points in the plane such that every planar 3-tree with $n$ vertices has a straight-line embedding in the plane in which the vertices are mapped to a subset of $S_n$. This is the first subquadratic upper bound on the cardinality of universal point sets for planar 3-trees, as well as for the class of 2-trees and serial parallel graphs.



## 1. Introduction

Every planar graph has a *straight-line embedding* in the plane [19] where the vertices are mapped to distinct points and the edges to pairwise noncrossing straight line segments between the corresponding vertices. A set $S \subset \mathbb{R}^2$ of points in the plane is called *n-universal* if every $n$-vertex planar graph has a straight-line embedding in $\mathbb{R}^2$ such that the vertices are mapped to a subset of $S$. Similarly, $S \subset \mathbb{R}^2$ is *n-universal for a family* $\mathcal{G}$ of planar graphs if every $n$-vertex planar graph in $\mathcal{G}$ has a straight-line embedding in $\mathbb{R}^2$ such that the vertices are mapped to a subset of $S$. It is a longstanding open problem to determine the minimum size $f(n)$ of an $n$-universal point set for all $n \in \mathbb{N}$. Our main result is that there is an $n$-universal point set of size $O(n^{3/2} \log n)$ for the class of planar graphs of treewidth at most three.

**Theorem 1.** *For every $n \in \mathbb{N}$, there is an $n$-universal point set of cardinality $O(n^{3/2} \log n)$ for planar 3-trees.*

A graph is called a *k-tree*, for some $k \in \mathbb{N}$, if it can be constructed by the following iterative process: start with a $k$-vertex clique and successively add new vertices such that each new vertex has exactly $k$ neighbors that form a clique in the current graph. For example, 1-trees are the same as trees; 2-trees are maximal series-parallel graphs, and include also all outerplanar graphs. In general, $k$-trees are the maximal graphs with treewidth $k$. A planar 3-tree is a 3-tree that is planar. Theorem 1 is the first subquadratic upper bound on the size of $n$-universal point sets for planar 3-trees, for 2-trees, and for series-parallel graphs.

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*Related previous work.* In a pivotal paper, de Fraysseix, Pach and Pollack [11] showed that an $n$-universal set must have at least $n + (1 - o(1))\sqrt{n}$ points. Chrobak and Karloff [10] improved the lower bound to $1.098n$ and later Kurowski [24] to $(1.235 - o(1))n$. This is the currently known best lower bound for $n$-universal sets in general. De Fraysseix et al. [11] and Schnyder [26] independently showed that there are $n$-universal sets of size $O(n^2)$. In fact, an $(n-1) \times (n-1)$ section of the integer lattice is $n$-universal [9,26] for every $n \geq 3$. Alternatively, a $\frac{4}{3}n \times \frac{2}{3}n$ section of the integer lattice is also $n$-universal [6]. The quadratic upper bound is the best possible if the point set is restricted to sections of the integer lattice: Frati and Patrignani [22] showed (based on earlier work by Dolev et al. [13]) that if a rectangular section of the integer lattice is $n$-universal, then it must contain at least $n^2/9 + \Omega(n)$ points. Currently the smallest $n$-universal point sets were obtained by Bannister et al. [2]. Their cardinality is $\frac{n^2}{4}$ and they are not rectangular sections of the integer lattice, instead, the result uses pattern avoiding permutations.

Grid drawings have been studied intensively due to their versatile applications. It is known that sections of the integer lattice with $o(n^2)$ points are $n$-universal for certain classes of graphs. For example, Di Battista and Frati [12] proved that an $O(n^{1.48})$ size integer grid is $n$-universal for *outerplanar* graphs. Frati [21] showed that 2-trees on $n$ vertices require a grid of size at least $\Omega(n2^{\sqrt{\log n}})$. Biedl [3] observed that the grid embedding of all $n$-vertex 2-trees requires an $\Omega(n) \times \Omega(n)$ section of the integer lattice *if* the combinatorial embedding (i.e., all vertex–edge and edge–face incidences) is given. On the other hand, Zhou et al. [27] showed recently that every $n$-vertex series-parallel graph, and thus, every 2-tree, has a straight-line embedding in a $\frac{2}{3}n \times \frac{2}{3}n$ section of the integer lattice and a section of the integer lattice of area $0.3941n^2$. Researchers have studied classes of planar graphs that admit $n$-universal point sets of size $o(n^2)$. A classic result in this direction, due to Gritzmann et al. [23] (see also [5]), is that every set of $n$ points in general position is $n$-universal for *outerplanar graphs*. Angelini et al. [1] generalized this result and showed that there exists an $n$-universal point set of size $O(n(\log n/\log\log n)^2)$ for so-called *simply nested* planar graphs. A planar graph is simply nested if it can be reduced to an outerplanar graph by successively deleting chordless cycles from the boundary of the outer face. Recently, Bannister et al. [2] found $n$-universal point sets of size $O(n \log n)$ for simply nested planar graphs, and $O(n \text{ polylog } n)$ for planar graphs of bounded pathwidth.

Theorem 1 provides a new broad class of planar graphs that admit subquadratic $n$-universal sets.

Algorithmic questions pertaining to the straight-line embedding of planar graphs have also been studied. The *point set embeddability* problem asks whether a given planar graph $G$ has a straight-line embedding such that the vertices are mapped to a given point set $S \subset \mathbb{R}^2$. The problem is known to be NP-hard [8], and remains NP-hard even for 3-connected planar graphs [15], triangulations and 2-connected outerplanar graphs [4]. However, it has a polynomial-time solution for 3-trees [16,17,25]. In a *polyline* embedding of a plane graph, the edges are represented by pairwise noncrossing polygonal paths. Biedl [3] proved that every 2-tree with $n$ vertices has a polyline embedding where the vertices are mapped to an $O(n) \times O(\sqrt{n})$ section of the integer lattice, and each edge is a polyline with at most two bends. Everett et al. [18] showed that there is a set $S_n$ of $n$ points in the plane, for every $n \in \mathbb{N}$, such that every $n$-vertex planar graph has a polyline embedding onto $S_n$ and at most one bend per edge. Dujmović et al. [14] constructed a point set $S'_n$ of size $O(n^2/\log n)$ for every $n \in \mathbb{N}$ such that every $n$-vertex planar graph has a polyline embedding with at most one bend per edge in which the vertices as well as all the bend points of the edges are mapped to $S'_n$.

*Organization.* We briefly review some structural properties of planar 3-trees (Section 2), then construct a point set $S_n \subset \mathbb{R}^2$ for every $n \in \mathbb{N}$ (Section 3), and show that it is $n$-universal for planar 3-trees (Section 4).

## 2. Basic properties of planar three-trees

A graph $G$ is a *planar 3-tree* if it can be constructed by the following iterative procedure. Initially, let $G = K_3$, the complete graph with three vertices. Successively augment $G$ by adding one new vertex $u$ and three new edges that join $u$ to three vertices of a triangle such that no two vertices are connected to all the vertices of the same triangle. A planar 3-tree can be embedded in the plane simultaneously with the iterative process: the initial triangle forms the outer-face and each new vertex $u$ is inserted in the interior of the face corresponding to the triangle it is attached to.

The iterative augmentation process that produces a 3-tree $G$ can be represented by a rooted tree $T = T(G)$ as follows (this is called a *face-representative tree* in [20]). Refer to Fig. 1. The nodes of $T$ correspond to the triangles of $G$. For convenience we denote a vertex of $T$ by its corresponding triangle in $G$. The root of $T$ corresponds to the initial triangle of $G$. When $G$ is augmented by a new vertex $u$ connected to the vertices of the triangle $\Delta = v_1v_2v_3$, we attach three new leaves to $\Delta$ corresponding to the triangles $v_1v_2u$, $v_1uv_3$ and $uv_2v_3$. For a node $\Delta$ of $T$, let $T_\Delta$ denote the subtree of $T$ rooted at $\Delta$. Let $V_\Delta$ denote the set of vertices of $G$ embedded in the interior of $\Delta$.

In Section 4, we embed the vertices of a planar 3-tree on a point set by traversing the tree $T$ from the root. The initial triangle $abc$ will be the outer face in the embedding such that the edge $ab$ is a horizontal line segment, and the vertex $c$ is the top vertex (i.e., it has maximal $y$-coordinate). We then successively insert the remaining $n-3$ vertices of $G$, each of which subdivides a triangular face into three triangles. We label the vertices of each triangle of $G$ as *left*, *right* and *top* vertex, respectively. These labels are assigned (without knowing the specifics of our embedding algorithm) as follows. Label the three vertices of the initial triangle in $G$ arbitrarily as *left*, *right* and *top*, respectively. When $G$ is augmented by a new vertex $u$ and edges $uv_1$, $uv_2$, and $uv_3$, where $v_1$ is the left, $v_2$ is the right, and $v_3$ is the top vertex of an existing triangle $v_1v_2v_3$, then let $v_1$, $v_2$, and $v_3$ keeps their labels left, right, top, respectively) in the new triangles $v_1v_2u$, $v_2v_3u$