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On the complexity of determining the irregular chromatic index of a graph

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ABSTRACT

An undirected simple graph G is locally irregular if adjacent vertices of G have different degrees. An edge-colouring ϕ of G is locally irregular if each colour class of ϕ induces a locally irregular subgraph of G. The irregular chromatic index $\chi'_{irr}(G)$ of G is the least number of colours used by a locally irregular edge-colouring of G (if any). We show that the problem of determining the irregular chromatic index of a graph can be handled in linear time when restricted to trees, but it remains NP-complete in general.

1. Introduction

How to distinguish the vertices of some undirected simple graph G? One natural way to proceed consists in considering the degrees of the vertices of G, namely to consider that any two vertices are distinguished whenever they have distinct degrees. But distinguishing via the degrees is not relevant in general as it can be easily proved that every simple graph with order at least 2 necessarily has two vertices with the same degree.

To overcome this issue, Chartrand et al. proposed the following approach [5]: transform the graph G into some totally irregular *multigraph* G' by replacing each edge e of G by a set of n_e parallel edges, with $n_e \geq 1$. Since two vertices are adjacent in G' if and only if they are adjacent in G, the structures of G and G' are similar. In that case, we are interested in finding such a multigraph G' which minimizes the quantity $\max\{n_e \,/\, e \in E(G)\}$.

This problem can be expressed as an edge-weighting problem as follows. Let $w : E(G) \to \{1, \dots, k\}$ be a k-edge-weighting of G. For each vertex $v \in V(G)$, define

$$c_w(v) := \sum_{vu \in E(G)} w(vu)$$

as the sum of the weights "incident" to v. If c_w is injective, that is $c_w(u) \neq c_w(v)$ for every two distinct vertices u and v of G, we say that w is *vertex-sum-distinguishing*. Note that if w is vertex-sum-distinguishing, the multigraph G' obtained from G by replacing each edge e of G by $w(e)$ parallel edges is totally irregular. Regarding the problem introduced above, we are interested in finding a vertex-sum-distinguishing edge-weighting of G that minimizes the number k of weights. The

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smallest k for which a graph G admits a vertex-sum-distinguishing k-edge-weighting is called the *irregularity strength* of G in the literature.

The notion of vertex-sum-distinguishing edge-weighting of graphs gave birth to dozens of variants (see e.g. [1,9,11], or [7] for a complete survey dedicated to this topic). One such variant, considered by Karoński, Łuczak and Thomason in [9], is defined as follows. A k-edge-weighting w is *neighbour-sum-distinguishing* if $c_w(u) \neq c_w(v)$ for every two *adjacent* vertices of G. The multigraph G' obtained from G by replacing each edge e of G by $w(e)$ parallel edges is now *locally irregular*, in the sense that only adjacent vertices are distinguished by their degrees. Analogous notions to the one of local irregularity can also be found in the literature, see e.g. [3].

In [10], Nierhoff proved that graphs with large order n have irregularity strength at most $n-1$, this upper bound being tight. In contrast, Karoński, Łuczak and Thomason conjectured in [9] that for every graph with no isolated edge one can produce a neighbour-sum-distinguishing 3-edge-weighting:

1-2-3 Conjecture. *Every graph with no isolated edge admits a neighbour-sum-distinguishing 3-edge-weighting.*

We refer the interested reader to [12] for an up-to-date survey dedicated to the 1-2-3 Conjecture. It is still not known whether the 1-2-3 Conjecture is true for regular graphs in general. One way for dealing with this question is to consider the following edge-colouring notion. An (improper) edge-colouring ϕ of G is *locally irregular* if every colour class of ϕ induces a locally irregular subgraph of G. As pointed out in [4], a locally irregular 2-edge-colouring of a regular graph G is also a neighbour-sum-distinguishing 2-edge-weighting of G. Hence, studying locally irregular 2-edge-colourings of graphs can be a way to tackle the 1-2-3 Conjecture in the context of regular graphs.

For a given graph G, we are interested in finding the least number of colours needed by a locally irregular edge-colouring of G (if any), called the *irregular chromatic index* of G and denoted by $\chi'_{irr}(G)$. If G does not admit any locally irregular edge-colouring, we say that G is *non-colourable* and let $\chi'_{irr}(G) = \infty$. It was shown in [4] that a graph G is non-colourable if and only if G is either an odd-length path, or an odd-length cycle, or a “tree-like” graph obtained by connecting an arbitrary number of triangles in a specific way. Due to their simple structure, such graphs can be recognized in polynomial time.

All known colourable graphs have irregular chromatic index at most 3, and graphs with irregular chromatic index exactly k are known for every $k \in \{1, 2, 3\}$. For instance, we have $\chi'_{irr}(P_3) = 1$, $\chi'_{irr}(P_{2q+1}) = 2$ for every $q \geq 2$, and $\chi'_{irr}(C_{2q'}) = 3$ for every odd $q' \geq 3$, where, for every $n \geq 1$, P_n and C_n denote the path and the cycle on n vertices, respectively. The following conjecture was proposed in [4]:

Local-Irregularity Conjecture. *Every colourable graph has irregular chromatic index at most 3.*

The Local-Irregularity Conjecture was verified for several classes of graphs in [4]. In particular, we have the following result (recall that a tree is non-colourable if and only if it is an odd-length path):

Theorem 1.1. *(See Baudon et al. [4].) If T is a colourable tree, then $\chi'_{irr}(T) \leq 3$.*

If the Local-Irregularity Conjecture turned out to be true, then all colourable graphs would have irregular chromatic index 1, 2 or 3. Therefore, a natural question is to find out whether it is easy to determine the irregular chromatic index of a given graph. This leads to the following decision problem:

Locally-Irregular k-Edge-Colouring
Instance: A graph G.
Question: Do we have $\chi'_{irr}(G) \leq k$?

Clearly, $\chi'_{irr}(G) = 1$ if and only if G is itself locally irregular. Since checking whether a graph is locally irregular can be done in polynomial time, Locally-Irregular 1-Edge-Colouring is in P. If the Local-Irregularity Conjecture were true, then any colourable graph would have irregular chromatic index less than k for every $k \geq 3$ and, for such a value of k, the problem Locally-Irregular k-Edge-Colouring would thus be equivalent to the problem of determining whether G is colourable (which is easy, as noticed above). Hence, if the Local-Irregularity Conjecture were true, then we would get that Locally-Irregular k-Edge-Colouring is in P for every $k \geq 3$.

In this paper, we investigate the status of the remaining problem Locally-Irregular 2-Edge-Colouring. We will prove in Section 3 that Locally-Irregular 2-Edge-Colouring is easy when restricted to trees, and in Section 4 that Locally-Irregular 2-Edge-Colouring is NP-complete in general. More precisely, we will prove the following:

Theorem 1.2. *There is a linear-time algorithm for solving* Locally-Irregular 2-Edge-Colouring *when restricted to trees.*

Theorem 1.3. Locally-Irregular 2-Edge-Colouring *is NP-complete in general, even when restricted to planar graphs with maximum degree at most* 6.