Journal of Discrete Algorithms 26 (2014) 1–6

Contents lists available at ScienceDirect

Journal of Discrete Algorithms

www.elsevier.com/locate/jda

On the metric dimension of HDN

Dacheng Xu, Jianxi Fan *

School of Computer Science and Technology, Soochow University, Suzhou 215006, China

ARTICLE INFO

Article history:
Received 23 March 2012
Received in revised form 17 September 2013
Accepted 18 September 2013
Available online 2 October 2013

Keywords:
Metric basis
Metric dimension
HDN1(n)
HDN2(n)

ABSTRACT

The concept of metric basis is useful for robot navigation. In graph G, a robot is aware of its current location by sending signals to obtain the distances between itself and the landmarks in G. Its position is determined uniquely in G if it knows its distances to sufficiently many landmarks. The metric basis of G is defined as the minimum set of landmarks such that all other vertices in G can be uniquely determined and the metric dimension of G is defined as the cardinality of the minimum set of landmarks. The major contribution of this paper is that we have partly solved the open problem proposed by Manuel et al. [9], by proving that the metric dimension of HDN1(n) and HDN2(n) are either 3 or 4. However, the problem of finding the exact metric dimension of HDN networks is still open.

1. Introduction

If we treat a robot moving from one position to another in Euclidean space as a point, then we can study the navigation in a graph [6,8]. In a graph G, a robot is aware of its current location by sending signals to obtain the distances between itself and the landmarks in G. Obviously, for any graph G, it is significant to find out the position and the minimum number of landmarks such that all other vertices in G can be uniquely determined by their distances to the landmarks. In fact, the set composed of these landmarks is the metric basis of G, and the cardinality of this set is the metric dimension of G. The concept of metric basis is also used in chemistry [3,6] and combinatorial optimization [11].

Khuller et al. [8] proved that the metric dimension of path and d-dimensional grid is 1 and d, respectively. They also showed that the problem of deciding whether the metric dimension of G is less than or equal to k is NP-complete. Chartrand et al. [3] studied the metric dimension of complete graphs K_n, complete bigraphs $K_{m,n}$, trees, unicyclic graphs, and ordinary connected graphs. Fehr et al. [5] considered the metric dimension of Cayley digraphs. Javaid et al. [7] investigated the metric dimension of a family of circulant graphs $C_n(1,2)$, and Imran et al. [6] extended the research of [7]. They proved the metric dimension of $C_n(1,2,3)$ is 4 when $n \equiv 2,3,4,5 \pmod 6$ and gave an upper bound of the metric dimension of $C_n(1,2,3)$ when $n \equiv 0,1 \pmod 6$. Other related research results about the metric dimension appeared in [1,13,14].

Stojmenovic [12] proposed a family of variants of meshes and tori, which includes honeycomb meshes, honeycomb tori, and others. Compared with 2D-mesh and tori, honeycomb networks have better topological properties. The degree, diameter, and bisection width of honeycomb mesh are 3, $1.63\sqrt{N}$, and $0.82\sqrt{N}$, respectively, where N is the number of vertices. The hexagonal mesh [4,10] is the dual of honeycomb mesh. Manual et al. [9] proved the metric dimension of honeycomb mesh is 3 by proving that the metric dimension of hexagonal mesh is 3. They also introduced two hex derived networks: HDN1(n) and HDN2(n), where $n \geqslant 2$. They have some superior performances over honeycomb meshes. The diameter of HDN1(n) and HDN2(n) are both $0.67\sqrt{N}$, less than honeycomb meshes. Manual et al. [9] proposed an open problem to show that the

* Corresponding author.
E-mail addresses: xudacheng06@163.com (D. Xu), jxfan@suda.edu.cn (J. Fan).

1570-8667/$ – see front matter
http://dx.doi.org/10.1016/j.jda.2013.09.002

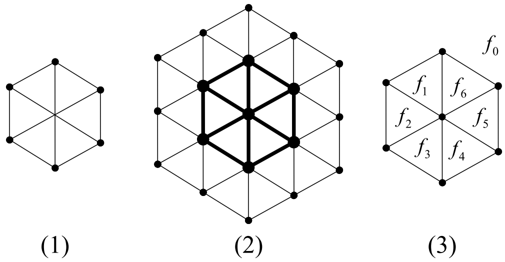

Fig. 1. Hexagonal meshes: (1) HX(2), (2) HX(3), and (3) all faces in HX(2).

metric dimension of HDN1(n) and HDN2(n) are between 3 and 5. In this paper, we have partly solved this open problem with a more tight bound that the metric dimension of HDN1(n) and HDN2(n) are either 3 or 4.

This work is organized as follows. Section 2 provides the definitions of metric basis, metric dimension, hexagonal network, HDN1(n), and HDN2(n). Detailed proofs that the metric dimension of HDN1(n) and HDN2(n) are either 3 or 4 are given in Section 3. We make a conclusion in Section 4.

2. Preliminaries

For an undirected graph $G = (V(G), E(G))$, $V(G)$ is the set of vertices and $E(G)$ is the set of edges. For two vertices u and $v \in V(G)$, the distance between u and v is $d(u, v)$ which is defined as the length of a shortest $u - v$ path in G. Clearly, $d(u, v) = d(v, u)$. An r-neighborhood of v, denoted by $N_r(v)$, is a subset U of $V(G)$, which means the distance between every vertex in U and v is r, i.e. $N_r(v) = \{u \in V(G) \mid d(u, v) = r\}$. Clearly, if $u \in N_{r1}(v) \cap N_{r2}(w)$, then $d(u, v) = r1$ and $d(u, w) = r2$. The degree of vertex v is denoted by $\deg(v)$.

If $W \subset V(G)$ such that for any two vertices u and $v \in V(G)$ there exists a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$, then W is a **locating set** of G. The vertices in W are called locating landmarks. A locating set containing a minimum number of landmarks is a **minimum locating set** of G, which is called a **metric basis** of G. The cardinality of the metric basis is called the **metric dimension**, which is denoted by $\mathrm{md}(G)$.

Chen et al. [4] proposed the **hexagonal mesh**. A hexagonal mesh is made up with a set of triangles as shown in Fig. 1. 1-dimensional hexagonal mesh does not exist. A 2-dimensional hexagonal mesh HX(2) is composed of six triangles (see Fig. 1(1)). A 3-dimensional hexagonal mesh HX(3) is obtained from HX(2) by adding a layer of triangles around the boundary of HX(2) (see Fig. 1(2)). Similarly, HX(n) is obtained from HX($n-1$) by adding a layer of triangles around the boundary of HX($n-1$).

A plane graph G partitions the rest of the plane into a number of arcwise-connected open sets, which are called the **faces** of G [2]. If two faces are adjacent, then they have at least one common edge. Every plane graph has one and only one unbounded face, called the **outer face**. Fig. 1(3) shows that HX(2) has seven faces $f_0, f_1, \dots, f_6$ where f_0 is the outer face and f_1 is adjacent to f_0, f_2 and f_6.

In plane graph HX(n), suppose any arbitrary face f is bijective to one vertex f^* except the outer face. If f^* is located in the face f and we connect the vertices of f with f^*, then we get HDN1(n). Fig. 2(1) demonstrates HDN1(3). Assume that f is adjacent to $f_1, f_2, \dots, f_k$ and $f_1^*, f_2^*, \dots, f_k^*$ are bijective to $f_1, f_2, \dots, f_k$, respectively. If we join the vertices of f and $f_1^*, f_2^*, \dots, f_k^*$ with f^*, then we get HDN2(n). Clearly, HDN1(n) is a subgraph of HDN2(n). Fig. 2(2) shows HDN2(3). HDN1(n) and HDN2(n) are collectively known as HDN(n), where $n \geqslant 2$.

In [9], a convenient coordinate system for HX(n) was introduced. Actually, the coordinate system applies for HDN(n) too (see Fig. 2), where x, y and z axes parallel to three edge directions of the hexagon. Lines that parallel to the coordinate axes x, y and z are x-lines, y-lines, and z-lines, respectively. All vertices on the x-lines have the same x-coordinate, all vertices on the y-lines have the same y-coordinate, and all vertices on the z-lines have the same z-coordinate. Thus, every vertex in HDN(n) is assigned a single coordinate (x, y, z). In Fig. 2, the coordinate of A, B, C, D, and E are $(3, 6, 3)$, $(-3, -6, -3)$, $(-3, 3, 6)$, $(-6, -3, 3)$, and $(6, 3, -3)$, respectively.

For HDN(n), $n \geqslant 2$, let $\gamma = (3(n-1), 3(n-1), 0)$, $\beta = (3(n-1), 0, -3(n-1))$, $\alpha = (0, -3(n-1), -3(n-1))$, $\eta = (-3(n-1), -3(n-1), 0)$, $\sigma = (-3(n-1), 0, 3(n-1))$, $\mu = (0, 3(n-1), 3(n-1))$, and $O = (0, 0, 0)$ (see Fig. 3).

For any other fundamental graph theoretical terminology, please refer to [2].

3. The metric dimension of HDN

In this section, we firstly prove $\mathrm{md}(\mathrm{HDN}(n)) \geqslant 3$. Then, we give a locating set W of HDN such that $|W| = 4$. Finally, we come to a conclusion $3 \leqslant \mathrm{md}(\mathrm{HDN}(n)) \leqslant 4$.

Suppose that W is a locating set of G. If $u \in V(G)$, $v \in W$, and $u \neq v$, then $d(u, v) > d(v, v)$. Thus, if we want to prove $W = \{w_1, w_2, \dots, w_k\}$ is a locating set of G, we only need to prove that for any two vertices $u, v \in V(G) \backslash W$, there is a w_i such that $d(u, w_i) \neq d(v, w_i)$, $1 \leqslant i \leqslant k$.