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# Dominating induced matchings in graphs without a skew star

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ABSTRACT

We study the problem of determining whether a graph $G$ has an induced matching that dominates every edge of the graph, which is also known as EFFICIENT EDGE DOMINATION. This problem is known to be NP-complete in general graphs, but it can be solved in polynomial time for graphs in some special classes, such as weakly chordal, $P_7$-free or claw-free graphs. In the present paper we extend the polynomial-time solvability of the problem from claw-free graphs to graphs without a skew star, where a skew star is a tree with exactly three vertices of degree 1 being of distance 1, 2, 3 from the only vertex of degree 3.



## 1. Introduction

We study the problem of determining whether the vertices of a graph can be partitioned into two subsets $B$ and $W$ so that $B$ induces a graph of vertex degree 1 (also known as an induced matching) and $W$ induces a graph of vertex degree 0 (i.e. an independent set). Throughout the paper we call the vertices of $B$ black and the vertices of $W$ white and say that a graph partitionable into an induced matching and an independent set admits a black–white partition. This problem appears in the literature under various names, such as EFFICIENT EDGE DOMINATION [3,5,12,18,19] or DOMINATING INDUCED MATCHING [2,4,6,13], and finds applications in various fields, such as parallel resource allocation of parallel processing systems [16] and encoding theory and network routing [12]. This problem also has relations to some other algorithmic graph problems, such as 3-COLOURABILITY and MAXIMUM INDUCED MATCHING. In particular, it is not difficult to see that every graph admitting a black–white partition is 3-colourable. Also, in [5] it was shown that if a graph admits a black–white partition, then the black vertices form an induced matching of maximum size.

From an algorithmic point of view, the DOMINATING INDUCED MATCHING problem is difficult, i.e. it is NP-complete [12]. Moreover, it remains difficult under substantial restrictions. For instance, in [14] it was shown that the problem is NP-complete for cubic graphs, and in [5] this result was extended to $d$-regular graphs for an arbitrary $d \geqslant 3$. The problem was also shown to be NP-complete for bipartite graphs [18] and planar bipartite graphs [19]. The NP-completeness results for bounded degree graphs and for bipartite graphs have been strengthened in [4] as follows.

Denote by $\mathcal{S}_k$ the class of $(C_3, \ldots, C_k, H_1, \ldots, H_k)$-free bipartite graphs of vertex degree at most 3, where $C_k$ is a chordless cycle on $k$ vertices and $H_k$ is the graph represented in Fig. 1. Associate with every graph $G$ a parameter $\kappa(G)$, which is the maximum $k$ such that $G \in \mathcal{S}_k$. If $G$ belongs to no class $\mathcal{S}_k$, then $\kappa(G)$ is defined to be 0, and if $G$ belongs to all classes $\mathcal{S}_k$, then $\kappa(G)$ is defined to be $\infty$. Finally, for a set of graphs $M$, define $\kappa(M) = \sup\{\kappa(G)\colon\ G \in M\}$.

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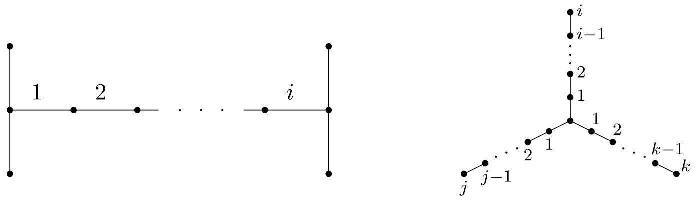


**Fig. 1.** Graphs $H_i$ (left) and $S_{i,j,k}$ (right).

**Theorem 1.** *(See [4].) Let $M$ be a set of graphs and $X$ the class of $M$-free bipartite graphs of vertex degree at most 3. If $\kappa(M) < \infty$, then the* DOMINATING INDUCED MATCHING *problem is NP-complete in the class $X$.*

Unless $P = NP$, Theorem 1 provides a necessary condition for polynomial-time solvability of the problem in classes of graphs defined by forbidden induced subgraphs. In particular, given a set $M$ of forbidden graphs, the problem is polynomial-time solvable in the class of $M$-free graphs only if $\kappa(M) = \infty$. Three basic ways to unbind the parameter $\kappa$ is to include in the set $M$ of forbidden graphs

(1) arbitrarily large cycles,
(2) arbitrarily large graphs of the form $H_k$,
(3) a graph $G$ with $\kappa(G) = \infty$.

Nearly all polynomial-time results available in the literature deal with graph classes of the first type. This includes bipartite permutation graphs [18], convex graphs [13], chordal graphs [19] and hole-free graphs [3]. Nothing is known about the complexity of the problem in classes of the second type and only two results are available for classes of the third type. By definition, $\kappa(G) = \infty$ if and only if $G$ belongs to all classes $\mathcal{S}_k$, i.e. $G$ belongs to the intersection $\bigcap \mathcal{S}_k$ taken over all possible values of $k$. Let us denote this intersection by $\mathcal{S}$. Thus, according to Theorem 1, if $M$ is a finite set, then the problem is polynomial-time solvable in the class of $M$-free graphs only if $M$ contains a graph from $\mathcal{S}$. We believe that the converse is also true and formally state this as a conjecture.

**Conjecture 1.** *Let $M$ be a finite set of graphs. Unless $P = NP$, the* DOMINATING INDUCED MATCHING *problem is polynomial-time solvable in the class of $M$-free graphs if and only if $M$ contains a graph from $\mathcal{S}$.*

Clearly, to prove the conjecture it is sufficient to consider finite sets $M$ consisting of a single graph $G$ that belongs to $\mathcal{S}$. It is not difficult to see that $G \in \mathcal{S}$ if and only if every connected component of $G$ is of the form $S_{i,j,k}$ (see Fig. 1). The smallest non-trivial graph of this form is $S_{1,1,1}$, also known as a claw.

The class of claw-free graphs received much attention in the literature due to many attractive properties of graphs in this class, see for example [7,11,15,20]. In particular, in [20] Minty develops a polynomial-time algorithm for the MAXIMUM INDEPENDENT SET problem in claw-free graphs, which extends the celebrated Edmonds' solution for the MAXIMUM MATCHING problem [10]. Recently [17], this solution was further extended to the class of $S_{1,1,2}$-free graphs.

The class of claw-free graphs is also easy for the DOMINATING INDUCED MATCHING problem [6], which is one of the two polynomially solvable cases of type 3 for the problem. The second solvable case deals with $P_7$-free graphs, as was recently proved in [2] (note that $P_7 = S_{0,3,3}$). In the present paper, we extend the solution for claw-free graphs to $S_{1,2,3}$-free graphs. Throughout the paper we call $S_{1,2,3}$ a skew star.

The organisation of the paper is as follows. In the rest of this section we introduce basic notations and terminology. In Section 2 we describe a number of useful reductions that help solve the problem. In Sections 3 we solve the problem for $S_{1,2,2}$-free graphs and then in Section 4 extend the solution to $S_{1,2,3}$-free graphs.

For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. If $v \in V(G)$, then $N(v)$ is the neighbourhood of $v$, i.e. the set of vertices adjacent to $v$. The degree of $v$ is $|N(v)|$. A graph is $k$-regular if the degree of each vertex is $k$. An independent set in $G$ is a subset of pairwise non-adjacent vertices. For a subset $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of $G$ induced by vertices of $U$. If a graph $G$ does not contain induced subgraphs isomorphic to a graph $H$, we say that $G$ is $H$-free and call $H$ a forbidden induced subgraph for $G$. As usual, $K_n$ is the complete graph on $n$ vertices, and $C_n$ ($P_n$) is the chordless cycle (path) on $n$ vertices. Diamond and butterfly are two graphs represented in Fig. 2.

## 2. Preliminaries

We view the DOMINATING INDUCED MATCHING problem as the problem of colouring the vertices of a graph with two colours, black and white, so that no white vertex has a white neighbour and every black vertex has exactly one black neighbour. Assigning one of the two possible colours to the vertices of $G$ will be called *colouring* of $G$. A colouring is *partial* if only part of the vertices of $G$ are assigned colours, otherwise it is *total*. In a partial colouring, a black vertex that has a black neighbour is called *matched*. A partial colouring is *valid* if