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Steiner tree reoptimization in graphs with sharpened triangle inequality ☆,☆☆

Hans-Joachim Böckenhauer, Karin Freiermuth, Juraj Hromkovič, Tobias Mömke, Andreas Sprock, Björn Steffen *

Department of Computer Science, ETH Zurich, Switzerland

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ABSTRACT

In this paper, we deal with several reoptimization variants of the Steiner tree problem in graphs obeying a sharpened β-triangle inequality. A reoptimization algorithm exploits the knowledge of an optimal solution to a problem instance for finding good solutions for a locally modified instance. We show that, in graphs satisfying a sharpened triangle inequality (and even in graphs where edge-costs are restricted to the values 1 and $1+\gamma$ for an arbitrary small $\gamma > 0$), Steiner tree reoptimization still is NP-hard for several different types of local modifications, and even APX-hard for some of them.

As for the upper bounds, for some local modifications, we design linear-time $(1/2+\beta)$-approximation algorithms, and even polynomial-time approximation schemes, whereas for metric graphs ($\beta = 1$), none of these reoptimization variants is known to permit a PTAS. As a building block for some of these algorithms, we employ a 2β-approximation algorithm for the classical Steiner tree problem on such instances, which might be of independent interest since it improves over the previously best known ratio for any $\beta < 1/2 + \ln(3)/4 \approx 0.775$.

1. Introduction

The Steiner tree problem is a very prominent optimization problem with many practical applications, especially in network design, see for example [18,19]. Given a complete weighted graph $G = (V, E)$ with edge cost function c and a set $S \subseteq V$ of vertices called terminals, the Steiner tree problem consists of finding a minimum-cost connected subgraph of G containing all vertices from S. The problem is known to be APX-hard, even if the edge costs are restricted to 1 and 2 [4]. A minimum spanning tree on the terminal vertices (w.r.t. the metric closure of the edge costs) is sufficient for achieving a 2-approximation (see, e.g., [19]), and the best currently known approximation ratio for the Steiner tree problem is $1 + \ln(3)/2 \approx 1.55$ for general edge costs and 1.28 for edge costs 1 and 2 [20].

In this paper, we analyze the hardness of even more restricted input instances. More precisely, we consider all instances where the edge costs are restricted to the values 1 and $1+\gamma$ for any $0 < \gamma$. Also this restricted problem variant is known to be APX-hard [17]. In particular, restricting the edge costs in the described way also implies the same hardness results for the class of Steiner tree problems where the edge-costs satisfy the sharpened β-triangle inequality, i.e., where the cost function c satisfies the condition $c(\{v_1, v_2\}) \leqslant \beta \cdot (c(\{v_1, v_3\}) + c(\{v_3, v_2\}))$, for some $1/2 \leqslant \beta < 1$ and for all vertices v_1,

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* Corresponding author.

E-mail addresses: hjb@inf.ethz.ch (H.-J. Böckenhauer), fkarin@inf.ethz.ch (K. Freiermuth), jhromkov@inf.ethz.ch (J. Hromkovič), tmoemke@inf.ethz.ch (T. Mömke), asprock@inf.ethz.ch (A. Sprock), steffenb@inf.ethz.ch (B. Steffen).

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v_2, and v_3. The graphs satisfying a sharpened triangle inequality form a subclass of the class of all metric graphs. Intuitively speaking, for vertices that are points in the Euclidean plane, a parameter value $\beta < 1$ prevents that three vertices can be placed on the same line. For more details and motivation of the sharpened triangle inequality, see [9].

To analyze how the transition from metric Steiner graphs ($\beta = 1$) to Steiner graphs with sharpened β-triangle inequality influences the computational hardness, we consider the question whether additional knowledge about the input is helpful for finding a good solution. More precisely, we consider the model of reoptimization algorithms which handles problems where an instance together with one of its optimal solutions is given and the problem is to find a good solution for a locally modified instance. This concept of reoptimization was mentioned for the first time in [21] in the context of postoptimality analysis for a scheduling problem. Since then, the concept of reoptimization has been investigated for several different problems like the traveling salesman problem [1,3,10,8], knapsack problems [2], covering problems [7], and the shortest common superstring problem [6]. In these papers, it was shown that, for some problems, the reoptimization variant is exactly as hard as the original problem, whereas reoptimization can help a lot for improving the approximation ratio for other problems. For an overview of some results see also [11]. These results show that the reoptimization concept gives new insight into the hardness of the underlying optimization problems and allows for a more fine-grained complexity analysis.

The Steiner tree reoptimization problem in general weighted graphs was previously investigated in [5,12,14] for various types of local modifications. We show that eight reoptimization variants (insertion and deletion of terminal or non-terminal vertices, increasing and decreasing edge costs, and changing the status of vertices from terminal to non-terminal and vice versa) are NP-hard on graphs with edge costs restricted to 1 and $1 + \gamma$. The best approximation algorithms for the four reoptimization variants considered in [5] (a terminal becomes a non-terminal or vice versa; the cost of an edge increases or decreases) achieve a constant approximation ratio in metric graphs. Here, we show that, on β-metric graphs, all of these four cases permit, in contrast to the non-reoptimization problem, a PTAS for any $\beta < 1$. When the local modification, however, consists in removing vertices, we show that the Steiner tree reoptimization is as hard to approximate as the original problem.

The two algorithmically most interesting reoptimization variants are the addition of terminal and of nonterminal vertices. For these modifications, we prove the APX-hardness of the corresponding reoptimization variants, which solves also the analogous open problem for reoptimization in Steiner trees with arbitrary edge costs. Escoffier et al. [14] designed simple linear-time algorithms for metric input instances ($\beta = 1$) which achieve an approximation ratio of 3/2. Using the same algorithms, but a much more complex and technically involved analysis, we prove a $(1/2 + \beta)$-approximation for graphs satisfying a sharpened β-triangle inequality. Note that the ratio $(1/2 + \beta)$ tends to 3/2 for β tending to 1 and to 1 for β tending to 1/2. These proofs employ a 2β-approximation algorithm for the classical non-reoptimization version of the Steiner tree problem in β-metric graphs which may be of independent interest since it improves over the previously best known ratio for any $\beta < 1/2 + \ln(3)/4 \approx 0.775$.

2. Preliminaries

Given a graph $G = (V, E)$ and a subset $S \subseteq V$ of vertices, called *terminals*, a *Steiner tree* for (G, S) is a subtree T of G spanning all terminals, i.e., $T = (V(T), E(T))$ is a tree such that $S \subseteq V(T) \subseteq V$ and $E(T) \subseteq E$. The vertices in $V - S$ are called *non-terminals*.

In a weighted graph $G = (V, E)$ with cost function $c : E \to \mathbb{Q}^+$, a *minimum Steiner tree* is a Steiner tree T of minimum cost, i.e., minimizing $\sum_{e\in E(T)} c(e)$ over all Steiner trees of G. In the remainder of the paper, we denote by $G = (V, E)$ a complete, undirected edge-weighted graph with a cost function c. (Missing edges are considered to be edges with the cost of the shortest path between the corresponding vertices.) The vertex set V of G is also denoted by $V(G)$ and the edge set E of G is also denoted by $E(G)$. Furthermore, we denote by $S \subseteq V(G)$ the set of terminals of G. The sum of the costs of all edges in a subgraph H of G is defined by $c(H) = \sum_{e\in E(H)} c(e)$. For the cost of an edge $\{x, y\}$, we use the notation $c(x, y)$ instead of $c(\{x, y\})$.

We are now ready to define the underlying optimization problem for our further investigations.

The *minimum Steiner tree problem* (*Min-STP*) in connected edge-weighted graphs is the problem of finding a minimum Steiner tree for an input instance (G, S, c). If the cost function c satisfies the β-triangle inequality, the minimum Steiner tree problem on the input instance (G, S, c) is called *Min-Δ_β-STP*. Similar to the Min-Δ_β-STP, we consider the problem *Min-$(1, 1+\gamma)$-STP*, where only edges of cost 1 and $1+\gamma$ are allowed. The relation of the Min-Δ_β-STP and the Min-$(1, 1+\gamma)$-STP is as follows.

Lemma 1. *For any graph $G = (V, E)$ and any $0 < \gamma$, any cost function $c : E \to \{1, 1 + \gamma\}$ satisfies the $(1 + \gamma)/2$-triangle inequality.*

Proof. Let x, y, and z be three different vertices in V. If all three edges $\{x, y\}$, $\{y, z\}$, and $\{z, x\}$ cost the same, then obviously our claim holds. Otherwise, either two of the edges have cost 1 or two of the edges have cost $1 + \gamma$. Since

$$1 + \gamma \leqslant \frac{1+\gamma}{2} \cdot (1 + 1) \leqslant \frac{1+\gamma}{2} \cdot (1 + 1 + \gamma) \leqslant \frac{1+\gamma}{2} \cdot (1 + \gamma + 1 + \gamma),$$

our claim also holds for these cases. □