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On the negative cost girth problem in planar networks

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ABSTRACT

In this paper, we discuss an efficient divide-and-conquer algorithm for the negative cost girth (NCG) problem in planar networks. Recall that the girth of an unweighted graph (directed or undirected) is the length of the shortest cycle in the graph. We extend the notion of girth to arbitrarily weighted networks as follows: The negative cost girth of a graph is the number of edges in a shortest negative cost simple cycle, i.e., a negative cost cycle having the fewest number of edges. The NCG problem in general networks has been well-studied, and there exist several algorithms for this problem. Clearly, the extant algorithms for the NCG problem can be used when the input network is restricted to be planar. However, the techniques used in this paper result in an algorithm with a superior running time. Our algorithm is based on the well-known Lipton–Tarjan planar separator theorem. On a directed, weighted, planar network $\mathbf{G} = \langle \mathbf{V}, \mathbf{E}, \mathbf{c} \rangle$ with n vertices, $m = O(n)$ edges, and negative cost girth k, the algorithm runs in $O(n^{1.5} \cdot k)$ time. This is a significant improvement over the $O(m \cdot n \cdot k) = O(n^2 \cdot k)$ running time that results, when the fastest known topology-oblivious algorithm for the NCG problem is restricted to planar networks. Additionally, our algorithm can be extended to find the NCG in selected generalizations of planar networks.

1. Introduction

This paper presents a new divide-and-conquer algorithm for the negative cost girth (NCG) problem in planar networks. The girth of an unweighted graph is defined as the length (i.e., number of edges) of the shortest cycle. If the network is acyclic, then the girth is infinity. In this paper, we extend the notion of girth to arbitrarily weighted, directed graphs (i.e., networks) and introduce the notion of negative cost girth in a network. Briefly, the negative cost girth of a network is the length of a negative cost cycle with the fewest number of edges. The NCG problem finds applications in several domains, such as constraint-solving, program verification and real-time scheduling [24,26]. This problem has been well studied in general networks. The first polynomial time algorithm for this problem was proposed in [27] and runs in $O(n^3 \cdot \log k)$ time, where n is the number of vertices, and k is the NCG.

In this paper, we focus on determining the negative cost girth in planar, directed networks. Existing NCG algorithms for general networks can indeed be used to solve the NCG problem in planar networks. However, the extant algorithms are

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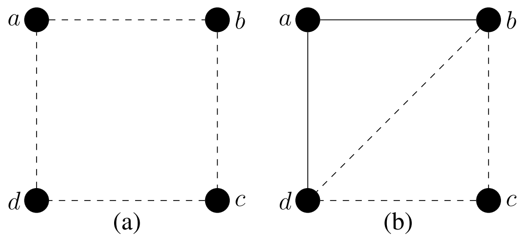

Fig. 1. Girth examples for undirected networks. The network in (a) has girth 4, while the network in (b) has girth 3.

topology-oblivious. We show that the property of planarity can be exploited to design an algorithm that is asymptotically superior to any previously known NCG algorithm, even when restricted to planar networks.

An application of Farkas' Lemma establishes that finding a negative cost cycle in a network is equivalent to proving the unsatisfiability of the corresponding difference constraint system (DCS) [5]. This means that any negative cost cycle can serve as a certificate for verifying the unsatisfiability of the DCS. However, end users prefer small-sized certificates and the negative cost cycle having the fewest number of edges is clearly the smallest-size certificate of unsatisfiability.

The NCG problem is motivated by a number of applications, as discussed in [27]. In program verification, an important subclass of satisfiability modulo theories (SMT) solvers are those devoted to difference logic (or difference constraint logic), which refers to the satisfiability of an arbitrary boolean combination of difference constraints [25]. In real-time scheduling, timing relationships are often circular in nature. Such relationships are represented by a constraint network [12]. If the constraints are infeasible, then there must exist a negative cost cycle [26]. Typically, the infeasibility is usually the result of a small infeasible subset [18]. Consequently, the goal of real-time designers is to find these small infeasible subsets and relax the corresponding constraints.

Our emphasis on planar networks in this paper arises from two distinct domains, viz., VLSI design and graph visualization. In both these cases, the underlying network model must be planar.

Suppose we are given a weighted, planar network $\mathbf{G} = \langle \mathbf{V}, \mathbf{E}, \mathbf{c} \rangle$ with n vertices, m edges, cost function $\mathbf{c} : \mathbf{E} \to \mathbb{Z}$ and negative cost girth k. The current fastest topology-oblivious NCG algorithm runs in $O(m \cdot n \cdot k)$ time [28]. In a planar network, we have $m \in O(n)$. Thus, for planar networks, the algorithm in [28] takes $O(n^2 \cdot k)$ time. However, our NCG algorithm for planar networks runs in $O(n^{1.5} \cdot k)$ time, which is a substantial improvement. Furthermore, our algorithm can be extended to work on a generalization of planar networks.

The principal contributions of this paper are as follows:

1. An $O(m \cdot k)$ time algorithm for finding the NCG in general networks, assuming that the network has an NCG of k and that we are provided at least one vertex in an NCG cycle.
2. An $O(n^{1.5} \cdot k)$ time algorithm for finding the NCG in planar networks.

The rest of this paper is organized as follows: Section 2 provides a formal description of the problem. We discuss approaches for related problems in the literature in Section 3. Section 4 presents a strategy for finding the NCG in general networks, when we are given at least one vertex in an NCG cycle. Section 5 discusses how this strategy is exploited to determine the NCG in planar networks. We analyze the time complexity and prove the correctness of the algorithm in Section 6. Section 7 describes how we can extend our algorithm to generalizations of planar networks. We detail our conclusions in Section 8 by summarizing our contributions and discussing avenues for future research.

2. Statement of problem

Let $\mathbf{G} = \langle \mathbf{V}, \mathbf{E}, \mathbf{c} \rangle$ denote a simple, directed network, where $\mathbf{V}$ is the vertex set with n vertices, $\mathbf{E}$ is the edge set with m edges, and $\mathbf{c} : \mathbf{E} \to \mathbb{R}$ is the cost function that assigns a real number to each edge in $\mathbf{E}$. The cost of edge e_{ij} is denoted by c_{ij}.

The *girth* of an undirected, unweighted graph is defined as the length, or number of edges, of the shortest simple cycle in the network. If the graph is acyclic, then the girth of the graph is infinity. For example, the graph in Fig. 1(a) has girth 4, and the graph in Fig. 1(b) has girth 3. The cycles representing the girth are indicated in dashed lines.

In this paper, we extend the notion of girth to directed networks with arbitrarily weighted edges. We define the *negative cost girth* of a network as the length (i.e., number of edges) of a negative cost simple cycle having the fewest number of edges. Any negative cost cycle whose length is the NCG is termed as an *NCG cycle*. If the network does not have any negative cost cycles, then the negative cost girth of the network is infinity.

In Fig. 2(a), the NCG of the network is clearly 4. In Fig. 2(b), we have two cycles. The length of one cycle is 3, and the length of the other cycle is 4. By definition, the girth of the network is 3. However, the total cost of the corresponding cycle is not negative. The other cycle is a negative cost cycle, and there are no other negative cost cycles with fewer edges. Therefore, the NCG of the network is 4. An NCG cycle for each network is indicated in dashed lines.