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On stable cutsets in claw-free graphs and planar graphs [☆]

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Abstract

A stable cutset in a connected graph is a stable set whose deletion disconnects the graph. Let K_4 and $K_{1,3}$ (claw) denote the complete (bipartite) graph on 4 and $1+3$ vertices. It is NP-complete to decide whether a line graph (hence a claw-free graph) with maximum degree five or a K_4-free graph admits a stable cutset. Here we describe algorithms deciding in polynomial time whether a claw-free graph with maximum degree at most four or whether a (claw, K_4)-free graph admits a stable cutset. As a by-product we obtain that the stable cutset problem is polynomially solvable for claw-free planar graphs, and also for planar line graphs.

Thus, the computational complexity of the stable cutset problem is completely determined for claw-free graphs with respect to degree constraint, and for claw-free planar graphs. Moreover, we prove that the stable cutset problem remains NP-complete for K_4-free planar graphs with maximum degree five.

Keywords: Stable cutset; Matching-cut; Claw-free graph; Planar graph

1. Introduction

In a graph, a *stable set* (a *clique*) is a set of pairwise non-adjacent (adjacent) vertices. A *cutset* (or *separator*) of a graph G is a set S of vertices such that $G - S$ is disconnected. A *stable cutset* (a *clique cutset*) is a cutset which is also a stable set (a clique).

Clique cutsets are a well-studied kind of separators in the literature, and have been used in divide-and-conquer algorithms for various graph problems, such as graph colouring and finding maximum stable sets; see [24,27]. Applications of clique cutsets in algorithm designing based on the fact that clique cutsets in arbitrary graphs can be found in polynomial time [2,13,24,26,27].

The importance of stable cutsets has been demonstrated first in [7,25] in connection to perfect graphs. TUCKER [25] proved that if S is a stable cutset in G and if no induced cycle of odd length at least five in G has a vertex in S

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then the colouring problem on G can be reduced to the same problem on the smaller subgraphs induced by S and the components of $G - S$.

Later, the papers [3–5,12,16,18] discussed the computational complexity and efficient solvability of STABLE CUTSET ("Does a given graph admit a stable cutset?"). It was shown in [18] that STABLE CUTSET is NP-complete on line graphs of bipartite graphs, hence on perfect graphs; see also Theorem 3 below.

Actually, stable cutsets (in line graphs) have been also studied under other notion. A graph is *decomposable* (cf. [14]) if its vertices can be coloured red and blue in such a way that each colour appears on at least one vertex but each vertex v has at most one neighbour having a different colour from v. In other words, a graph is decomposable if its vertices can be partitioned into two nonempty parts such that the edges connecting vertices of different parts form an induced matching, a *matching-cut*. Matching-cuts have been studied in [1,6,10,11,18,20,23]. The papers [8,23] pointed out an application of matching-cuts in graph drawing.

Decomposability relates to stable cutsets as follows (see Section 5 for the definition of the linegraph $\mathrm{L}(G)$ of a graph G):

Proposition 1. *(See [3].) If $\mathrm{L}(G)$ has a stable cutset, then G is decomposable. If G is decomposable and has minimum degree at least 2, then $\mathrm{L}(G)$ has a stable cutset.*

Chvátal [6] proved that recognising decomposable graphs is NP-complete, even for graphs with maximum degree four. Thus, in terms of stable cutsets in line graphs, Chvátal's result may be reformulated as follows:

Theorem 2. *(See Chvátal [6].)* STABLE CUTSET *is NP-complete, even if the input is restricted to line graphs with maximum degree six.*

Theorem 2 has been improved as follows, stating that the computational complexity of STABLE CUTSET with respect to degree constraint is completely solved for line graphs:

Theorem 3. *(See [18].)* STABLE CUTSET *remains NP-complete if restricted to line graphs (of bipartite graphs) with maximum degree five, and is polynomially solvable for line graphs of maximum degree at most four.*

In particular, STABLE CUTSET is NP-complete for claw-free graphs with maximum degree 5. In [18], it is shown that STABLE CUTSET is solvable in linear time for arbitrary graphs with maximum degree at most 3. The complexity of STABLE CUTSET for graphs with maximum degree 4 is still open.

In this paper, we will improve the second part of Theorem 3 to the larger class of claw-free graphs as follows: STABLE CUTSET can be solved in polynomial time for claw-free graphs of maximum degree at most 4. Thus, with respect to degree constraint, the computational complexity of STABLE CUTSET is completely solved for claw-free graphs.

In [3], it was shown that STABLE CUTSET is NP-complete for K_4-free graphs (notice that for K_3-free graphs, STABLE CUTSET becomes trivial). Our second result is that STABLE CUTSET can be solved in polynomial time for (claw, K_4)-free-graphs. As a by-product, we will show that STABLE CUTSET is polynomially solvable for claw-free planar graphs. In particular, STABLE CUTSET is polynomially solvable for planar line graphs.

Finally, we show that STABLE CUTSET remains NP-complete on planar graphs with maximum degree five.

2. Preliminaries

Let G be a graph. The vertex set and the edge set of G is denoted by $V(G)$ and $E(G)$ respectively. Unless specified, we assume $|V(G)| = n$ and $|E(G)| = m$. The neighbourhood of a vertex v in G, denoted by $N_G(v)$, is the set of all vertices in G adjacent to v; if the context is clear, we simply write $N(v)$. Set $\deg(v) = |N(v)|$, the degree of the vertex v. For a subset $W \subseteq V(G)$, $G[W]$ is the subgraph of G induced by W and $G - W$ denotes the graph $G[V(G) \setminus W]$; if $W = \{w\}$, then we simply write $G - w$.

When discussing the computational complexity of STABLE CUTSET, we clearly may assume that

$$\text{no vertex } v \text{ in } G \text{ has a stable neighbourhood } N(v), \tag{1}$$