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Normal forms and normal theories in conditional rewriting

Salvador Lucas [a,b,*], José Meseguer [a]

[a] CS Dept., University of Illinois at Urbana-Champaign, United States
[b] DSIC, Universitat Politècnica de València, Spain

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ABSTRACT

We present several new concepts and results on conditional term rewriting within the general framework of order-sorted rewrite theories (OSRTs), which support types, subtypes and rewriting modulo axioms, and contains the more restricted framework of conditional term rewriting systems (CTRSs) as a special case. The concepts shed light on several subtle issues about conditional rewriting and conditional termination. We point out that the notions of *irreducible* term and of *normal form*, which coincide for unconditional rewriting, have been conflated for conditional rewriting but are in fact *totally different* notions. Normal form is a *stronger* concept. We call any rewrite theory where all irreducible terms are normal forms a *normal* theory. We argue that normality is essential to have good executability and computability properties. Therefore we call all other theories *abnormal*, freaks of nature to be avoided. The distinction between irreducible terms and normal forms helps in clarifying various notions of strong and weak termination. We show that abnormal theories can be terminating in various, equally abnormal ways; and argue that any computationally meaningful notion of strong or weak conditional termination should be a property of normal theories. In particular we define the notion of a *weakly operationally terminating* (or *weakly normalizing*) OSRT, discuss several evaluation mechanisms to compute normal forms in such theories, and investigate general conditions under which the rewriting-based operational semantics and the initial algebra semantics of a confluent, weakly normalizing OSRT coincide thanks to a notion of *canonical term algebra*. Finally, we investigate appropriate conditions and proof methods to ensure that a rewrite theory is normal; and characterize the stronger property of a rewrite theory being operationally terminating in terms of a natural generalization of the notion of quasi-decreasing order.

1. Introduction

This paper presents several new contributions to conditional term rewriting and to the semantics of declarative, rewriting-based languages. Conditional rewriting is considered within the general and highly expressive framework of *order-sorted rewrite theories* (OSRTs), that is, theories $\mathcal{R} = (\Sigma, B, R)$, where (Σ, B) is an order-sorted equational theory [17,7] with equational axioms B, and R is a collection of conditional rewrite rules with oriented conditions of the form: $\ell \to r \Leftarrow s_1 \to t_1, \ldots, s_n \to t_n$, which are applied *modulo* B. All the results are *in particular* new results for *Conditional Term Rewriting Systems* (CTRSs); that is, for order-sorted rewrite theories of the special form $\mathcal{R} = (\Sigma, \varnothing, R)$, with $B = \varnothing$ and Σ

* Corresponding author.
E-mail address: slucas@dsic.upv.es (S. Lucas).

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unsorted, i.e., having a *single* sort. The point of using OSRTs is, of course, that not only the CTRS-based syntactic rewriting, but the more general rewriting modulo axioms B is thus supported; and that, for obvious reasons of expressiveness, all well-known rule-based languages, e.g., [8,23,6,3], support not only rewriting modulo axioms, but also types and, often, subtypes. Therefore, the greater generality of OSRTs is not a caprice, but an absolute necessity for making formal specification and declarative programming expressive and practical.

Our contributions consist in asking and providing detailed answers to the following, innocent-sounding questions:

1. What is the right notion of *normal form* for an OSRT?
2. What is the right notion of *weak operational termination* for an OSRT?
3. Under what conditions can OSRTs be used as *declarative programs* having a well-behaved semantics? How can we *execute* such programs? How can their executability conditions be checked in practice?
4. Under what conditions does a confluent OSRT have a *canonical term algebra* that can be effectively computed and that provides a *complete agreement* between the operational semantics of the OSRT as a functional program, and its mathematical, initial algebra semantics?
5. Can the operational termination of OSRTs be characterized in terms of orders?

Surprisingly enough, some of these questions seem to never have been asked. At best, the issues involved seem to have remained implicit as not well-understood, anomalous features in the literature. Consider, for example, question (1) above, which asks about the notion of normal form. For unconditional term rewriting the notion is absolutely clear and unproblematic: a normal form is a term t that is *irreducible*, that is, such that there is no t' with $t \to_{\mathcal{R}} t'$. To the best of our knowledge, all the CTRS literature is unanimous in identifying normal forms with irreducible terms also in the conditional case. That is, the terms “normal form” and “irreducible term” are used with the *same* meaning by all authors.[1] However, for an OSRT, and in particular for a CTRS, the notion of normal form is actually highly problematic. The big problem is that for an OSRT there can be terms t that are irreducible in the above sense, i.e., there is no t' with $t \to_{\mathcal{R}} t'$, but such that when we give t to a rewrite engine for evaluation such an engine loops! For a trivial example, consider the single conditional rewrite rule $a \to b \Leftarrow a \to c$. Since the rewrite relation defined by this conditional rule is the empty set, the constant a is trivially irreducible; but the proof tree associated to the normalization of a using the CTRS inference system is *infinite* [13], and a rewrite engine that tries to evaluate a will loop when trying to satisfy the rule's condition. Therefore, calling a a *normal form* is a very bad joke, since, intuitively, a term is considered to be a normal form if it is “fully normalized,” that is, if it is the *result* of fully evaluating some input term by rewriting; but this is precisely what a in the above example is *not*.

Our answer to this puzzle is to introduce a precise distinction (fully articulated in the paper) between irreducible terms and normal forms: every term in normal form is irreducible, but, as the above example shows, not every irreducible term is a normal form. We call an OSRT *normal*[2] iff every irreducible term is a normal form, and call it *abnormal* otherwise. Abnormal theories, like the one above, are hopeless for executability purposes and should be viewed as monsters in the menagerie of CTRSs and OSRTs.

Termination is quite a subtle issue for OSRTs in general and CTRSs in particular. The distinction between irreducible terms and normal forms helps in clarifying various notions of termination. We show that abnormal theories can be terminating in various, equally abnormal ways; and argue that any computationally meaningful notion of strong or weak conditional termination should be a property of normal theories. Many notions of conditional termination have been proposed (see e.g., [20]), but it is by now well-understood that the most satisfactory notion from a computational point of view is that of *operational termination* [13] (more on this later). Here we ask and answer several questions, further developing this notion. One question is (5) above. For the case of deterministic 3-CTRS we proved in [13] that operational termination is *equivalent* to the order-based notion of *quasi-decreasingness*. In Section 6 we generalize this result to a similar result characterizing operational termination of OSRTs in terms of an (axiom-compatible) term ordering.

Another related question is question (2), which could be more simply rephrased as follows: what is the right notion of weak termination/normalization for OSRTs? Although notions of weak CTRS termination go back at least to [2], as we further explain in Section 3.1 they can be very misleading; that is, they can violate one's most basic intuitions about what termination *means*. Our distinction between irreducible terms and normal forms shows that there are in fact *two* possible notions, a computationally ill-behaved one (*weak termination*: every term has a terminating rewrite sequence ending in an irreducible term), and a computationally well-behaved one (*weak operational termination*: every term has a normal form).

The notion of normal OSRT is closely related to question (3), namely, that of executability conditions for declarative, conditional rule-based programs, and their evaluation methods, i.e., their operational semantics. As we explain in Section 4,

[1] Nevertheless, in the seminal paper [2], for $\mathcal{R}$ a confluent and orthogonal CTRS a distinction is made between the set $\mathsf{Irr}(\mathcal{R})$ of its irreducible terms (called there “normal forms”), and a subset $\mathsf{Irr}_f(\mathcal{R}) \subseteq \mathsf{Irr}(\mathcal{R})$ of irreducible terms of “finite order.” However, as we further explain in Section 3.1, the notion of “irreducible term of finite order” is too weak to capture the intuitive notion of normal form, namely, a term that is the *result* of the term normalization process.

[2] Note that this meaning of “normal” is in open conflict with the definition of a *normal CTRS* (see, e.g., [20]) as a CTRS whose rewrite rules R are all of the form $l \to r$ if $\bigwedge_{i=1..n} u_i \to v_i$ with each v_j ground and, not only R-irreducible, but, furthermore, R_u-irreducible, where R_u is the set of unconditional rules obtained from R by dropping all conditions. Since: (i) the terminology “normal” in, e.g., [20], is not universally accepted (e.g., with an extra orthogonality condition they are instead called type III_n in [2]), and (ii) these two meanings of “normal” are so different, no confusion should arise.