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Relational style laws and constructs of linear algebra

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A B S T R A C T

We present a few laws of linear algebra inspired by laws of relation algebra. The linear algebra laws are obtained from the relational ones by replacing union, intersection, composition and converse by the linear algebra operators of addition, Hadamard product, composition and transposition. Many of the modified expressions hold directly or with minor alterations.
We also define operators that sum up the content of rows and columns. These share many properties with the relational domain and codomain operators returning a subidentity corresponding to the domain and codomain of a relation. Finally, we use the linear algebra operators to write axioms defining direct sums and direct products and we show that there are other solutions in addition to the traditional – in the relational context – injection and projection relations.

1. Introduction

This paper presents a collection of laws of linear algebra that are similar to corresponding laws of relation algebra. The starting point is the observation that matrices with 0, 1 entries only are relations. Let Q and R be such matrices. Then their Hadamard product $Q \cdot R$, i.e., their entrywise arithmetic multiplication, is their intersection. The standard addition $Q + R$ and composition (multiplication) $Q\,R$ are not quite the union and relational composition, but they are not so far from that. Transpose R^{T} and conjugate transpose $R^{\dagger}$ are exactly the converse of R, where, for a matrix A with complex numbers as entries, $(A^{\dagger})_{i,j} = (A_{j,i})^{\dagger}$, with $(x + y\mathrm{i})^{\dagger} = x - y\mathrm{i}$. Our aim is to study what happens when the relational operators of a relational law are replaced by the linear algebra operators, and what happens when arbitrary matrices are used instead of relations.

Our purpose is to augment the repertoire of point-free laws of linear algebra, an endeavour in the spirit of the work of Macedo and Oliveira [1–3]. Some, if not most, of these laws are already known, but nevertheless we feel the "relational twist" is worth exploring.

Section 2 presents the notation and some basic laws. Section 3 introduces domain-like operators. Sections 4 and 5 are about direct sums and direct products; in both cases, the linear algebra setting yields additional solutions compared with the relational setting; these additional solutions are obtained by composing the relational solutions with unitary

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transformations. We conclude in Section 6. We assume knowledge of the relational material that is used below, which can be found in [4,5]. There are numerous textbooks on linear algebra; see, e.g., [6].

The paper is an extension of [7], with additional results, proofs and examples, especially in Section 5 on direct products.

2. Basic laws

We consider finite matrices over the complex numbers. In the sequel, the term *relations* refers to matrices with 0, 1 entries only. Variables A, B, C denote arbitrary matrices, D a diagonal matrix, V a column vector and P, Q, R relations. Matrix *composition* is denoted by juxtaposition, as is customary in linear algebra. The other operators are arithmetic multiplication $\times$, matrix addition $+$, Hadamard product $\cdot$ (entrywise multiplication $(A \cdot B)_{i,j} = A_{i,j} \times B_{i,j}$), transpose $^\mathsf{T}$, conjugate transpose †, entrywise conjugation $A^\ddagger$ (i.e., $(A^\ddagger)_{i,j} = (A_{i,j})^\dagger$), identity matrix $\mathbb{I}$ and zero matrix $\mathbf{0}$ ($\mathbf{0}_{i,j} = 0$ for all i, j). For relations, they are union $\cup$, intersection $\cap$, composition ;, converse $^\smile$ and universal relation $\mathbb{T}$ ($\mathbb{T}_{i,j} = 1$ for all i, j). The size of a matrix with m rows and n columns is indicated by $m \leftrightarrow n$, occasionally as a subscript. The unary operators have precedence over the binary ones. The order of increasing precedence for the binary operators is $(+, \cup)$, $(\cdot, \cap)$, (composition, ;). The ordering on relations is denoted by $\subseteq$ and the pointwise ordering on real matrices by $\leqslant$, i.e., $A \leqslant B \Leftrightarrow (\forall i, j \mid A_{i,j} \leqslant B_{i,j})$.

Some laws satisfied by these operators follow.

$$\begin{array}{llll}
\text{(a)} & A \cdot B = B \cdot A & & \\
\text{(b)} & A^\dagger = A^{\mathsf{T}\ddagger} = A^{\ddagger\mathsf{T}} & A^\mathsf{T} = A^{\dagger\ddagger} = A^{\ddagger\dagger} & A^\ddagger = A^{\dagger\mathsf{T}} = A^{\mathsf{T}\dagger} \\
\text{(c)} & (A \cdot B)^\mathsf{T} = A^\mathsf{T} \cdot B^\mathsf{T} & (A \cdot B)^\dagger = A^\dagger \cdot B^\dagger & (A \cdot B)^\ddagger = A^\ddagger \cdot B^\ddagger \\
\text{(d)} & (AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T} & (AB)^\dagger = B^\dagger A^\dagger & (AB)^\ddagger = A^\ddagger B^\ddagger \\
\text{(e)} & A^{\mathsf{T}\mathsf{T}} = A^{\dagger\dagger} = A & & \\
\text{(f)} & \mathbb{I}^\mathsf{T} = \mathbb{I} & (\mathbb{T}_{m\leftrightarrow n})^\mathsf{T} = \mathbb{T}_{n\leftrightarrow m} &
\end{array} \tag{1}$$

Using the Hadamard product we can characterise relations algebraically as the set of matrices A satisfying $A \cdot A = A$. For a relation R, $R^\smile = R^\mathsf{T} = R^\dagger$.

The universal relation $\mathbb{T}$ is the neutral element of the Hadamard product, i.e., $A \cdot \mathbb{T} = A$. As is customary in the relational setting, the same symbol $\mathbb{T}$ may denote matrices of different size (and similarly for $\mathbf{0}$ and $\mathbb{I}$).

Using matrix composition on relations rather than relational composition gives a more “quantitative” result. Indeed, $(QR)_{i,j}$ is the number of paths from i to j by following Q and then R, rather than simply indicating whether there is a path or not. In particular, all entries of the matrix $\mathbb{T}_{l\leftrightarrow m}\mathbb{T}_{m\leftrightarrow n}$ are m, the size of the intermediate set (rows for the first matrix, columns for the second):

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}.$$

Relations in combination with the Hadamard product can be used to impose “shapes” on arbitrary matrices. For an arbitrary matrix A and a relation R, we say that A has *shape* R iff $A \cdot R = A$. By the Hadamard characterisation of relations, every relation then has its own shape. For a further instance, if R is univalent, then $A \cdot R$ is a matrix with at most one non-zero entry in each row; thus, A has at most one non-zero entry in each row iff it has shape R for some univalent relation R. Instead of univalent relations, one may use equivalence relations, difunctional relations, symmetric relations, etc. to impose shapes. The shape of a matrix is not unique: if A has shape Q and $Q \subseteq R$, then A also has shape R.

We define a *subshape* relation $\sqsubseteq$ by

$$A \sqsubseteq B \;\Leftrightarrow\; (\exists \text{ relation } R \mid A = B \cdot R). \tag{2}$$

Intuitively, A results from B by replacing some entries in B by 0.

Proposition 1.

1. *Every matrix has shape* $\mathbb{T}$*, while only* $\mathbf{0}$ *has shape* $\mathbf{0}$*.*
2. *If* B *has shape* R *and* A *is arbitrary then* $A \cdot B$ *has shape* R *as well.*
3. *If* A *and* B *have shape* R *then* $A + B$ *has shape* R *as well.*
4. *The set of matrices of shape* R *forms an ideal in the ring of all matrices under* $+$ *and* $\cdot$*.*
5. *If* A *has shape* R *and* B *has shape* S *then* $A \cdot B$ *has shape* $R \cdot S$*.*
6. $\sqsubseteq$ *is a partial order.*
7. *Pointwise,* $\sqsubseteq$ *can be formulated as follows:* $A \sqsubseteq B \Leftrightarrow (\forall i, j \mid A_{i,j} = 0 \vee A_{i,j} = B_{i,j})$*.*

Proof.

1. Immediate from the definitions.
2. By associativity of $\cdot$ and the definition of shape, $(A \cdot B) \cdot R = A \cdot (B \cdot R) = A \cdot B$.