Journal of Logical and Algebraic Methods in Programming 83 (2014) 169–179

Contents lists available at ScienceDirect

Journal of Logical and Algebraic Methods in Programming

www.elsevier.com/locate/jlamp

Discrete dualities for some algebras with relations

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ARTICLE INFO

Article history:
Available online 5 March 2014

Dedicated to Gunther Schmidt on the occasion of his 75th birthday.

Keywords:
Discrete dualities
Proximities
Contact algebras
Syllogistic structures

ABSTRACT

In this paper we present a unifying discrete framework for various representation theorems in the field of spatial reasoning. We also show that the universal and existential quantifiers of restricted scope used in first order languages and represented as binary relations in the syllogistic algebras considered by Shepherdson (1956) [1] may be studied in this framework.

1. Introduction

In this paper we present discrete dualities between some classes of Boolean algebras or their reducts, additionally endowed with a binary relation, and the appropriate classes of relational systems (frames). We consider two types of algebras with relations: those studied in connection with spatial reasoning where the relations describe relationships between space regions, and those considered in theories of Aristotelian syllogistic presented in [1] where the relations represent statements of the form "All (resp. some) *a* are *b*".

For an overview of recent developments in the region based theory of space and related topics we invite the reader to consult [2] or [3].

By a *discrete duality* [4] we mean a system $\langle \mathsf{Alg}, \mathsf{Frm}, \mathfrak{Cm}, \mathfrak{Cf} \rangle$ where Alg is a class of algebras, Frm is a class of frames, $\mathfrak{Cm} : \mathsf{Frm} \to \mathsf{Alg}$ is a mapping assigning to every frame X in Frm its *complex algebra* $\mathfrak{Cm}(X)$ in such a way that $\mathfrak{Cm}(X)$ belongs to Alg, $\mathfrak{Cf} : \mathsf{Alg} \to \mathsf{Frm}$ is a mapping assigning to every algebra L in Alg its *canonical frame* $\mathfrak{Cf}(L)$ in such a way that $\mathfrak{Cf}(L)$ belongs to Frm, and the following two representation theorems hold:

1. Representation theorem for algebras: Every L in Alg is embeddable into $\mathfrak{Cm}(\mathfrak{Cf}(L))$.
2. Representation theorem for frames: Every X in Frm is embeddable into $\mathfrak{Cf}(\mathfrak{Cm}(X))$.

In the literature there are some topological representation theorems for algebras of spatial reasoning, see for example [5,6], but no corresponding abstract frames are considered, only the structures which – in terms of the notions used in

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1 Ivo Düntsch gratefully acknowledges support by the Natural Sciences and Engineering Research Council of Canada.
2 Ewa Orłowska gratefully acknowledges partial support from the National Science Centre project DEC-2011/02/A/HS1/00395.

http://dx.doi.org/10.1016/j.jlap.2014.02.006
2352-2208/

the definition of discrete duality (dd-notions) – are counterparts to canonical frames endowed with a topology are introduced. In [7] a discrete representation theorem for Boolean proximity algebras is presented, but there is no a representation theorem for frames. In [1] a necessary and sufficient condition is given for obtaining a representation theorem for syllogistic algebras, saying that every such algebra is embeddable into the algebra which – in terms of dd-notions – is the complex algebra of a frame. In the present paper we develop discrete representations for some of those algebras, introduce the corresponding classes of frames, and prove representation theorems for them.

In Section 3 we present a discrete duality for Boolean algebras endowed with a proximity relation and their corresponding frames. Following [7], the canonical frames are constructed with prime filters – as usual in the Stone representation theorem for Boolean algebras.

In Section 4 a discrete duality for Boolean algebras with a contact relation and the appropriate frames is developed. In this case the canonical frames are constructed with clans as in [6].

In Section 5 a discrete representation theorem is presented for a syllogistic ∀-algebra with the relation representing the universal quantifier with a restricted scope as considered in [1].

In Section 6 a discrete representation theorem for ∀∃-frames is developed. The representation structure is built with an algebra with relations representing both the universal and existential quantifiers of restricted scope presented in [1].

2. Notation and first definitions

If R, S are binary relations on X, then $R;S = \{\langle a, c\rangle\colon (\exists b)[aRb \text{ and } bSc]\}$ is the *relational product of R and S*, and $R^{\smile} = \{\langle a, b\rangle\colon bRa\}$ is the *converse of R*.

A frame is a pair $\langle X, R\rangle$, where X is a nonempty set and R is a binary relation on X. Since the frames needed for establishing a discrete duality for Boolean algebras are just nonempty sets, in this paper we also allow for such a "degenerate" notion of frame. For a frame $\langle X, R\rangle$ we set $R(x) = \{y\colon xRy\}$, and define two operators on 2^X by

$$\langle R\rangle(A) = \bigl\{x \in X\colon (\exists y \in X)[xRy \wedge y \in A]\bigr\} = \bigl\{x \in X\colon R(x) \cap A \neq \emptyset\bigr\}, \tag{1}$$

$$[R](A) = \bigl\{x \in X\colon (\forall y \in X)[xRy \Rightarrow y \in A]\bigr\} = \bigl\{x \in X\colon R(x) \subseteq A\bigr\}. \tag{2}$$

The following properties of $\langle R\rangle$ and $[R]$ are well known, see [8]:

Lemma 2.1.

1. *$\langle R\rangle$ is a normal operator which distributes over $\cup$.*
2. *$\langle R\rangle$ and $[R]$ are dual to each other, i.e. $[R](A) = X \setminus \langle R\rangle(X \setminus A)$.*
3. *$\langle R\rangle$ is a closure operator if and only if $[R]$ is an interior operator if and only if R is reflexive and transitive.*

For the definition of closure and interior operator see [9].

If $\leqslant$ is a partial order on X, we call $x, y \in X$ *compatible*, if there is some z such that $x \leqslant z$ and $y \leqslant z$, otherwise, they are called *incompatible*. We can consider $\langle\leqslant\rangle$ and $[\leqslant]$ as topological operators of closure and interior. If $A \subseteq X$, we let $\uparrow A = \{y\colon (\exists x)[x \in A \text{ and } x \leqslant y]\}$. If $A = \{x\}$ we often just write $\uparrow x$ instead of $\uparrow\{x\}$. We define $\downarrow A$ analogously.

The *order topology $\tau_{\leqslant}$ on X generated by* $\leqslant$ (also called the *Alexandrov topology*) has the sets of the form $[\leqslant](A)$ as a basis for the open sets. Since $[\leqslant]$ is an interior operator, $\tau_{\leqslant}$ is well defined; indeed, $\tau_{\leqslant}$ is closed under arbitrary intersections. Since for each $x \in X$, $\uparrow x$ is the smallest open set containing x, we see that the set $\{\uparrow x\colon x \in X\} \cup \{\emptyset\}$ also is a basis for $\tau_{\leqslant}$. $\mathrm{RegCl}(X, \tau_{\leqslant})$, the collection of regular closed sets in this topology, consists of sets of the form $\langle\leqslant\rangle[\leqslant](Y)$ for $Y \subseteq X$.

The operations on a Boolean algebra are denoted by $\wedge$ (meet), $\vee$ (join), $-$ (complement), 0 (minimum), and 1 (maximum); in particular, if R is a binary relation on X, then $-R$ is its complement in the Boolean set algebra with universe $2^{X\times X}$. We reserve the symbol $\neg$ for negation in syllogistic structures.

3. A discrete duality of proximity algebras and frames

In this section we show how the representation of proximity algebras of [7] fits into our framework. In this context, we call a structure $\langle X, R\rangle$ a *proximity frame*, if X is a nonempty set and R is a binary relation on X.

Suppose that B is a Boolean algebra. A binary relation δ on B is called a *proximity* if it satisfies the following for all $a, b, c \in B$:

Prox_1. $a\,\delta\,b$ implies $a \neq 0$ and $b \neq 0$.
Prox_2. $a\,\delta(b \vee c)$ if and only if $a\,\delta\,b$ or $a\,\delta\,c$.
Prox_3. $(a \vee b)\,\delta\,c$ if and only if $a\,\delta\,c$ or $b\,\delta\,c$.