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Relational properties of sequential composition of coalgebras

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ABSTRACT

In this paper we define a sequential composition for arbitrary coalgebras in a Dedekind category. We show some basic algebraic properties of this operation up to bisimulation. Furthermore, we consider certain recursive equations and provide an explicit solution, i.e., a solution not based on an iterative process.

1. Introduction

Coalgebras, bisimulation, and coinduction play an important role in mathematics and computer science. One of the first examples of bisimulations appear in process calculi such as CSP or CCS [5,7]. Such processes can be modeled by coalgebras, and the behavioral equivalence is based on bisimulation. These calculi also introduce the basic operations of parallel composition, summation, and prefixing on processes. In a recent paper [2] a coalgebraic logic for deterministic Mealy machines that is sound and complete was presented.

In addition, models of modal logics give naturally rise to coalgebras based on the underlying transition relation of $\Box$ and $\Diamond$ [10]. Bisimulation and coinduction are used to reason about models, property preserving constructions, and relationship to other logics. For example, Van Benthem's characterization theorem shows that modal logic is the fragment of first-order logic that is invariant under bisimulation. Another example is given by the method of filtration that is used to show that the satisfiability problem of certain modal logics is decidable. It has been shown that filtration is based on a bisimilarity relation [13].

Recently there has been great interest in using coinduction and bisimulation to reason about lazy functional programming languages. The first motivation for this approach was given in [1]. In this paper the lazy lambda calculus has been defined and it was shown a bisimulation, called applicative bisimulation, is a congruence on the terms of this calculus. The interest in lazy language also started intensive research on coalgebraic specification. These specifications are based on observation operations instead of constructors, an approach that leads to algebraic specifications. For an intensive study of examples of coalgebraic specifications we refer to [6].

For additional examples and a more detailed overview of methods and applications of coalgebras, bisimulation, and coinduction we refer to [11,12].

In this paper we want to consider coalgebras in the context of Dedekind categories. These categories provide a suitable abstraction to reason about relation [8,9]. Therefore, a coalgebra is a relation $Q : S \to F(S)$ with an appropriate functor F. This generalizes processes, labeled transition systems, or other coalgebraic structures such as Kripke structures, in multiple

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ways. First of all, relations in Dedekind category need not to be relations in the classical sense, i.e., subsets of the cartesian product of sets. Lattice-valued or L-fuzzy relations as well as probabilistic relations are models of Dedekind categories. Among the huge class of additional examples there are even non-representable Dedekind categories, i.e., categories where the morphisms are not equivalent to any notion of a relation based on sets of pairs. The second generalization is given by the functor F. A coalgebra $Q : S \to F(S)$ has two aspects. It provides a transition from a state to a successor state together with an additional effect encoded by F. In this paper we will make no assumption on this additional behavior.

Relational coalgebras have been studied before in [17,18]. These paper mainly studied parallel composition of processes and its relational properties. It was shown, for example, that equivalence classes of certain coalgebras form an ordered category based on parallel composition. In the current paper we want to concentrate on sequential composition.

This paper is organized as follows. Section 2 introduce the basic mathematical notions such as Dedekind categories, relators, coalgebras, rooted coalgebras, and bisimulations. In Section 3 we define the sequential composition of two rooted coalgebras. After investigating some basic properties of this operation we define the sum of rooted coalgebras in Section 4. This is corresponds to the disjoint union of coalgebras by taking their roots into account. Furthermore, some basic property of this operation are shown. The main theorem of this section shows that the sequential composition distributes from the right over the sum. In Section 5 we consider simple recursion on a coalgebra by providing an explicit solution to the equation $X = P.X$ where , denotes the sequential composition. Finally, Section 6 outlines how to solve general equations in a language based on the construction defined in this paper. As in Section 5 the solution is given explicitly.

Due to length restrictions we had to omit several proofs in this paper. The missing parts can be found in the extended version of the paper [19].

2. Mathematical preliminaries

Throughout this paper, we use the following notation. To indicate that a morphism R of a category $\mathcal{R}$ has source A and target B we write $R : A \to B$. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$ and the composition of a morphism $R : A \to B$ followed by a morphism $S : B \to C$ by $R; S$. Last but not least, the identity morphism on A is denoted by $\mathbb{I}_A$.

In this section we recall some fundamentals on Dedekind categories [8,9]. This kind of category is called locally complete division allegories in [4].

Definition 1. A Dedekind category $\mathcal{R}$ is a category satisfying the following:

1. For all objects A and B the collection $\mathcal{R}[A, B]$ of morphisms/relations is a complete distributive lattice. Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap$, $\sqcup$, $\sqsubseteq$, $\perp\!\!\!\perp_{AB}$ and $\top\!\!\!\top_{AB}$, respectively.
2. There is a monotone operation $^{\smile}$ (called converse) mapping a relation $Q : A \to B$ to a relation $Q^{\smile} : B \to A$ such that $(Q; R)^{\smile} = R^{\smile}; Q^{\smile}$ and $(Q^{\smile})^{\smile} = Q$ for all relations $Q : A \to B$ and $R : B \to C$.
3. For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^{\smile}; S)$ holds.[2]
4. For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $Q : A \to B$ the following holds: $Q; R \sqsubseteq S \iff Q \sqsubseteq S/R$.

As already indicated in the definition above we will use morphism and relation interchangeably in the context of Dedekind categories.

Throughout this paper we will use several basic properties of Dedekind categories such as $\mathbb{I}_A^{\smile} = \mathbb{I}_A$, the monotonicity of composition in both parameters, or the distributivity of ; over $\sqcup$ without mentioning. For details we refer to [3,4,14–16].

An important class of relations within Dedekind categories are mappings.

Definition 2. Let $Q : A \to B$ be a relation. Then we call

1. Q univalent iff $Q^{\smile}; Q \sqsubseteq \mathbb{I}_B$,
2. Q total iff $\mathbb{I}_A \sqsubseteq Q; Q^{\smile}$,
3. Q a mapping iff Q is univalent and total.

In the next lemma we recall an important properties of mappings that we will use in this paper.

Lemma 1. *Suppose $Q : A \to B$ and $R : D \to C$ are relations, and $f : B \to C$ is a mapping. Then we have*

$$Q; f \sqsubseteq R \iff Q \sqsubseteq R; f^{\smile}.$$

[2] By convention the precedence of the operations decreases in the following order $.^{\smile}$ then ; then $\sqcap$.