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Completeness results for omega-regular algebras

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ABSTRACT

We present algebras that axiomatise the equational theories of two variants of omega-regular expressions. The first algebra simplifies a two-sorted infinitary axiomatisation by Wagner. Its completeness is established relatively to Wagner's completeness result. The second one is based on one-sorted algebras proposed by Park and Cohen. Its completeness is derived from our first result.

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1. Introduction

Regular algebras axiomatise the equational theory of regular expressions, that is, the congruence induced by regular language identity. They have been studied for several decades; completeness proofs for different variants have been given by Salomaa [12], Krob [8], Kozen [6], Boffa [2,3] and others.

Much less is known about ω-regular algebras, which expand the approach to ω-regular expressions and languages. It seems that the only published completeness result, for a two-sorted infinitary axiomatisation, is due to Wagner [15]. A one-sorted finitary alternative, called ω-algebra, has been proposed by Cohen [4]; similar axioms have been used previously by Park [10]. To our knowledge, completeness results for such simpler systems have not been published. Wilke [16] has proposed a two-sorted infinitary semigroup-based formalism for ω-regular languages in the context of the syntactic semigroup approach. This work, however, is not directly relevant to questions of completeness.

Wagner's axiomatisation expands Salomaa's regular algebra. He uses infinitary schemata to axiomatise the congruence corresponding to ω-regular language identity directly on ω-regular expressions. In the formation of X^ω, the empty word must be excluded from the regular language X. This is reflected by a recursively axiomatised proviso on regular expressions in his axioms as well as in Salomaa's. Kozen [6] has pointed out that 'the proviso [...] is not algebraic in the sense that it is not preserved under substitution. Consequently [one of Salomaa's axioms] is not valid under nonstandard interpretations [... and] must not be interpreted as a universal Horn formula'. This argument applies to Wagner's axioms as well. The question of providing complete finitary first-order axiomatisations of the equational theory of ω-regular expressions and languages therefore seems to be open.

Such axiomatisations are important since they admit models of computational interest beyond languages and may cover a wider range of applications. By linking these models with languages and automata, equational reasoning can become decidable. In the case of regular algebras, for instance, an identity holds in the regular language model if and only if it holds

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in the relational model. Hence both equational theories can be decided by automata. Completeness results for first-order ω-regular algebras promise similar benefits. The ω-operation as a notion of infinite iteration is important in total correctness semantics [9], refinement [14] or the analysis of reactive systems [4].

Our main contribution lies in two complete ω-regular algebras for two different kinds of ω-regular languages. These are obtained by considering signatures in which multiplicative units, which are interpreted by the empty word language, are absent. Our ω-regular algebras are therefore based on regular algebras in which a transitive closure operation rather than the Kleene star, which corresponds to the reflexive transitive closure, is axiomatised.

Our first algebra generalises Wagner's axiomatisation to finitely many quasi-identities without non-algebraic side conditions. Its regular reduct uses variants of Boffa's axioms which provide the weakest known complete regular algebras. We use an adjunction and embedding construction to establish completeness of this algebra with respect to the standard model of ω-regular languages and expressions relative to Wagner's completeness result. It follows that the ω-regular languages form the free algebras in the variety generated by this axiomatisation.

Our second algebra refines Cohen's one-sorted axiomatisation so that it faithfully captures the structure of ω-regular languages and expressions. We establish its soundness and completeness with respect to another notion of ω-regular language in which finite and infinite words can be mixed. The completeness proof is relative to our first result; it is based on a comparison of structural properties of languages in both models and their reflection through syntactic normal forms. It follows once more that these languages form the free algebras in the variety generated by the one-sorted axiomatisation.

We also study variants of additive units that arise in models of interest and prove completeness for these cases. This is motivated by the fact that the right annihilation law $X \cdot \emptyset = \emptyset$ holds in the first, but not in the second model of ω-languages. However, we show that this model cannot be captured entirely.

The remainder of this article is organised as follows. Section 2 recalls the standard notions of regular and ω-regular languages and expressions, Section 3 those of regular algebras. Section 4 presents a completeness result of a new variant of Boffa's regular algebras without multiplicative units with respect to regular languages of non-empty words. Section 5 introduces Wagner's algebras for ω-regular expressions and our finitary variant; in Section 6 we prove completeness of this variant relative to Wagner's result. Completeness results for algebras without additive units are presented in Section 7. Section 8 introduces our variant of Cohen's omega algebra; its soundness and completeness (without additive units) are proved in Sections 9 and 10; completeness with additive units is established in Section 11, although relative to a somewhat artificial model. Finally, Section 12 draws a conclusion.

2. Regular and omega-regular languages

Let Σ be an alphabet and let Σ^* denote the free monoid generated by Σ. The elements of Σ^* are *words* $a_1 \dots a_n$ with $a_i \in \Sigma$, including the *empty word* ε. A *language* is a subset of Σ^*. The *complex product*, the *Kleene star* and *Kleene plus* on languages $X, Y \subseteq \Sigma^*$ are defined as

$$X \cdot Y = \{xy \mid x \in X,\ x \in Y\}, \qquad X^* = \bigcup_{i \geq 0} X^i, \qquad X^+ = \bigcup_{i \geq 1} X^i,$$

where $X^0 = \{\varepsilon\}$ and $X^{i+1} = X \cdot X^i$. It follows that $X^* = \{\varepsilon\} \cup X^+$ and $X^+ = X \cdot X^*$. We usually drop multiplication symbols.

The *regular expressions* over Σ are defined by the grammar

$$\mathsf{Reg}_*(\Sigma) ::= 0 \mid 1 \mid a \in \Sigma \mid e + e \mid e \cdot e \mid e^*.$$

In addition, $s^+ = ss^*$. Let $L_* : \mathsf{Reg}_*(\Sigma) \to 2^{\Sigma^*}$ denote the homomorphism from regular expressions to languages defined by

$$L_*(0) = \emptyset, \qquad L_*(1) = \{\varepsilon\}, \qquad L_*(a) = \{a\}, \quad \text{for all } a \in \Sigma,$$
$$L_*(s + t) = L_*(s) \cup L_*(t), \qquad L_*(st) = L_*(s)L_*(t), \qquad L_*\big(s^*\big) = L_*(s)^*.$$

Then $L_*(s^+) = L_*(s)^+$. The *regular languages* over Σ are the homomorphic images of $\mathsf{Reg}_*(\Sigma)$ under L. We write $\mathsf{Reg}(\Sigma^*)$ for this set. Alternatively, regular languages are an inductive subset of Σ^* which contains $\emptyset$, $\{\varepsilon\}$ and the singleton sets $\{a\}$ for all $a \in \Sigma$, and which is closed under finite unions, finite language products and Kleene stars of regular sets.

Language identity induces a congruence on regular expressions,

$$s = t \quad \Leftrightarrow \quad L_*(s) = L_*(t),$$

where the equality sign has been overloaded. Axiomatising this congruence is the aim of *regular algebras*, which are the topic of the next section.

The function $o : \mathsf{Reg}_*(\Sigma) \to \mathsf{Reg}_*(\Sigma)$ defined by

$$o(0) = 0, \qquad o(1) = 1, \qquad o(a) = 0, \quad \text{for all } a \in \Sigma,$$
$$o(s + t) = o(s) + o(t), \qquad o(st) = o(s)o(t), \qquad o\big(s^*\big) = 1,$$

measures at the level of regular expressions whether a regular language contains the empty word:

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