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Multirelational representation theorems for complete idempotent left semirings

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ABSTRACT

Complete idempotent left semirings are a relaxation of quantales by giving up strictness and distributivity of composition over arbitrary joins from the left. It is known that the set of up-closed multirelations over a set forms a complete idempotent left semiring together with union, multirelational composition, the empty multirelation, and the membership relation. This paper provides a sufficient condition for a complete idempotent left semiring to be isomorphic to a complete idempotent left semiring consisting of up-closed multirelations, in which all joins, the least element, multiplication, and the unit element are respectively given by unions, empty multirelations, the multirelational composition, and the membership relation. Some equivalent conditions of the sufficient condition are also provided.

1. Introduction

This paper shows multirelational representation theorems for complete idempotent left semirings (IL-semirings for short). Complete IL-semirings are a relaxation of quantales by giving up strictness and distributivity of composition over arbitrary joins from the left. Quantales have been introduced by Mulvey [8] as complete join semilattices together with an associative composition satisfying the distributive laws. The same structures were also investigated by Conway [3] under the name of standard Kleene algebras (**S**-algebras). Multirelations are a generalisation of relations. These are studied as a semantic domain of programs [12] and game logic [11]. In the context of modal logics, multirelations appear as neighbourhood models or Scott–Montague models [2]. Up-closed multirelations are known as one of the typical models for complete idempotent left semirings [10].

Brown and Gurr [1] have shown that

- a quantale is isomorphic to a quantale consisting of binary relations, in which multiplication is given by the relational composition, and

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- a completely coprime algebraic quantale is isomorphic to a quantale consisting of binary relations, in which all joins with respect to the inclusion and multiplication are respectively given by unions and the relational composition.

Following the idea of [1], it is proved in [4] that

- a complete idempotent left semiring is isomorphic to a complete idempotent left semiring consisting of multirelations, in which multiplication is given by the multirelational composition, and
- a completely coprime algebraic complete idempotent left semiring is isomorphic to a complete idempotent left semiring consisting of multirelations, in which all joins with respect to the inclusion and multiplication are respectively given by unions and the multirelational composition.

The main result in [9] is that a powerset quantale is isomorphic to a quantale consisting of binary relations, in which all joins, the least element, multiplication, and the unit element are respectively given by unions, the empty relation, the relational composition, and the identity relations. This progress from [1] has become available by inducing relations on the set of completely coprime elements from a quantale instead of relations on the underlying set. Also, introducing invertibility for quantales, [9] has given a characterisation of powerset quantales.

Importing the point of observation from [9], this paper introduces powerset idempotent left semirings and shows that a powerset idempotent left semiring is isomorphic to a complete idempotent left semiring consisting of up-closed multirelations, in which all joins, the least element, multiplication, and the unit element are respectively given by unions, empty multirelations, the multirelational composition, and the membership relation. Moreover, introducing invertibility for complete idempotent left semiring, characterisations of complete coprime algebraic complete idempotent left semirings and powerset idempotent left semirings are given.

This paper is organised as follows. Section 2 provides definitions and examples of complete idempotent left semirings, multirelational complete idempotent left semirings, and generating sets for a complete idempotent left semiring. Notions of completely coprime algebraic complete idempotent left semirings and powerset idempotent left semirings are provided in Section 3. In Section 4 we define a function mapping elements of a complete idempotent left semiring to multirelations and show some properties of the function. Using the function, multirelational representation theorems for complete idempotent left semirings are exhibited in Section 5. Invertibility for complete idempotent left semirings is introduced in Section 6, and using this notion, other characterisations of completely coprime algebraic complete idempotent left semirings and powerset idempotent left semirings are given. Section 7 summarises this work.

2. Complete idempotent left semirings

Idempotent left semirings [7] are defined as follows.

Definition 2.1. An *idempotent left semiring*, or briefly an *IL-semiring* is a tuple $(S, +, \cdot, 0, 1)$ with a set S, two binary operations $+$ and $\cdot$, and $0, 1 \in S$ satisfying the following properties:

- $(S, +, 0)$ is an idempotent commutative monoid.
- $(S, \cdot, 1)$ is a monoid.
- For all $a, b, c \in S$, $a \cdot c + b \cdot c = (a + b) \cdot c$, $a \cdot b + a \cdot c \le a \cdot (b + c)$, and $0 \cdot a = 0$,

where the *natural order* $\le$ is given by $a \le b$ iff $a + b = b$.

We often abbreviate $a \cdot b$ to ab.

Remark 2.2. An IL-semiring S satisfying $ab + ac = a(b + c)$ and $a0 = 0$ for all $a, b, c \in S$ is an idempotent semiring.

Definition 2.3. A *complete* IL-semiring S is an IL-semiring satisfying the following properties: For each $A \subseteq S$,

- the least upper bound $\bigvee A$ of A exists in S, i.e., S is a complete join semilattice and
- $(\bigvee A)a = \bigvee\{xa \mid x \in A\}$ for each $a \in S$.

A complete IL-semiring preserving *right directed joins* is a complete IL-semiring satisfying

$$a\Big(\bigvee A\Big) = \bigvee\{ax \mid x \in A\}$$

for each element a and each non-empty directed subset A. A complete IL-semiring preserving *the right zero* is a complete IL-semiring satisfying $a0 = 0$ for each element a. A complete IL-semiring preserving *the right* $+$ is a complete IL-semiring satisfying $ab + ac = a(b + c)$ for any elements a, b, and c.