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Algebraic properties of stochastic effectivity functions

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Dedicated to Professor Prakash Panangaden on the occasion of his 60th birthday

ABSTRACT

Effectivity functions are the basic formalism for investigating the semantics game logic. We discuss algebraic properties of stochastic effectivity functions, in particular the relationship to stochastic relations, morphisms and congruences are defined, and the relationship of abstract logical equivalence and behavioral equivalence is investigated.

1. Introduction

This paper investigates some algebraic properties of stochastic effectivity functions, which have been used as the basic formalism for interpreting game logic, and which show some interesting relationships to non-deterministic labelled Markov processes and to stochastic relations, i.e., to Markov transition systems. Before entering into the discussion of the stochastic branch of this family of functions, it is interesting and illuminating to have a look at the evolution of these functions, so let us start with some historical remarks.

Historical remarks. Effectivity functions were first systematically investigated in the area of Social Choice, and here in particular for the modeling of voting systems [27, Chapter 7.2]. Moulin models the outcome of cooperative voting through a binary relation $\mathbf{E} \subseteq \mathcal{P}(N) \times \mathcal{P}(A)$ between coalitions of voters and subsets of outcomes, where a coalition is just a subset of the entire population. Here N is the set of voters, A the set of outcomes, and $\mathcal{P}$ denotes the power set. If $T\ \mathbf{E}\ B$, coalition T is said to be *effective for* the subset B of outcomes, thus coalition T can force an outcome in B. Among others, Moulin postulates that if T is effective for B, then $N \setminus T$ must not be effective for $S \setminus B$. Some examples (unanimity with status quo, veto functions) illustrate the approach.

Generalizing this in their work on Social Choice, Abdou and Keiding [1] define effectivity functions as special cases of conditional game forms. Such a *conditional game form* is a map $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{A} \setminus \{\emptyset\})$, where N and A are as above, $\mathcal{A}$ is a subset of $\mathcal{P}(A)$ with $\emptyset \in \mathcal{A}$, $A \in \mathcal{A}$. The family of all closed sets in a topological space or of all measurable subsets in a measurable space are mentioned as examples. If T is a coalition, $B \in E(T)$ models that coalition T can force an outcome in B. Thus the notion of effectivity is the same as in Moulin's proposal. Based on this, a first simple and general form of effectivity function is defined. A conditional game form $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{A} \setminus \{\emptyset\})$ is called an *effectivity function* iff $A \in E(T)$ for all non-empty coalitions T, and if $E(N) = \mathcal{A} \setminus \{\emptyset\}$. Thus a non-empty coalition can achieve something, and the community N has all options to choose from.

The functions discussed so far do not make an assumption on monotonicity; for the purposes of the present paper, however, monotone effectivity functions are of interest. This property appears to be natural, given the interpretation of

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effectivity functions: If an outcome in B can be forced by coalition T, then an outcome in each super set $B' \supseteq B$ can be forced by this coalition as well; if this is true for all coalitions, function E is called *monotone*.

The neighborhood relations used here are taken from the minimal models discussed in modal logics [4, Chapter 7.1], serving as basic mechanism for models which are more general than Kripke models. The association of the effectivity functions sketched here to a very similar notion investigated in economics is discussed in the survey paper [32, Section 2.3].

Game logic. Parikh [29], and later Pauly [30] propose interpreting game logic through a neighborhood model. Assign to each primitive game g and each player $\{1, 2\}$ a neighborhood relation $N_g^{(i)} \subseteq S \times \mathcal{P}(S)$ $(i = 1, 2)$ with the understanding that $sN_g^{(i)}X$ indicates player i having a strategy in state s to force a state in $X \subseteq S$. Here S is the set of states over which the game is interpreted. The fact that $sN_g^{(i)}X$ is sometimes described by saying that player i is effective for X (with game g in state s). It is desirable that $sN_g^{(i)}X$ and $X \subseteq X'$ imply $sN_g^{(i)}X'$ for all states s. We assume in addition that the game is *determined*, i.e., that exactly one of the players has a winning strategy. Thus $X \subseteq S$ is effective for player 1 in state s if and only if $S \setminus X$ is not effective for player 2 in that state. Consequently,

$$sN_g^{(2)}X \quad \Leftrightarrow \quad \neg\big(sN_g^{(1)}S \setminus X\big), \tag{1}$$

which in turn implies that we only have to cater for player 1. We will omit the superscript from the neighborhood relation N_g. Define a map $S \to \mathcal{P}(\mathcal{P}(S))$, again denoted by N_g, upon setting $N_g(s) := \{X \subseteq S \mid sN_gX\}$, then $N_g(s)$ is an upper closed subset of $\mathcal{P}(S)$ for all $s \in S$ from which relation N_g can be recovered. This function is called the *effectivity function* associated with relation N_g. From N_g another map $\breve{N}_g : \mathcal{P}(S) \to \mathcal{P}(S)$ is obtained upon setting

$$\breve{N}_g(A) := \{s \in S \mid sN_gA\} = \big\{s \in S \bigm| A \in N_g(s)\big\}.$$

Thus state s is an element of $\breve{N}_g(A)$ iff the first player has a strategy force the outcome in A when playing g in s. The operations on games can be taken care of through this family of maps, e.g., one sets recursively for the first player

$$\breve{N}_{g_1 \cup g_2}(A) := \breve{N}_{g_1}(A) \cup \breve{N}_{g_2}(A), \tag{2}$$

$$\breve{N}_{g_1;g_2}(A) := (\breve{N}_{g_1} \circ \breve{N}_{g_2})(A), \tag{3}$$

$$\breve{N}_{g^*} := \bigcup_{n \ge 0} \breve{N}_{g^n}(A), \tag{4}$$

with $g_1 \cup g_2$ denoting the game in which the first player chooses from games g_1, g_2, the game $g_1; g_2$ plays g_1 first, then g_2, and g^* is the indefinite iteration of game g. This refers only to player 1, player 2 is accommodated through $A \mapsto S \setminus \breve{N}_g(S \setminus A)$ by (1), since the game is determined. Pauly [30, Section 6.3] discusses the important point of determinacy of games and relates it briefly to the discussion in set theory [23, Section 33], [22, Section 12.3] or [24, Section 20].

The maps $\breve{N}_g$ serve in Parikh's original paper as a basis for defining the semantics of game logic. It turns out to be convenient for the present paper to go back, and to use effectivity functions as maps to upper closed subsets. When interpreting game logic probabilistically, however, constructions (2)–(4) are fairly meaningless, because one cannot talk about, e.g., the union of two probabilities. Hence another path had to be travelled, which may be seen as a technical generalization of the proposal [12] for interpreting propositional dynamic logic.

Requirements. The paper [11] proposes an approach through stochastic effectivity functions. This variant assigns to each state sets of probability distributions, to be specific, each state is assigned an upward closed set of measurable sets of these distributions. This property is inherited from the effectivity functions discussed above.

But let us have a look at other requirements for a stochastic effectivity function. They should generalize stochastic relations in the sense that each stochastic relation generates a stochastic effectivity function in a fairly natural way; this requirement will make it possible to consider the interpretation of modal logics through stochastic Kripke models as interpretations through stochastic effectivity functions, hereby enabling us to compare — or contrast, as the case may be — results for these formalisms. We want also to state that morphisms for stochastic relations are morphisms for the associated effectivity functions. In addition, we want to have some notion of measurability for these functions. Consequently, the set of all distributions has to be made a measurable space. This raises immediately the question of measurability of the effectivity function proper, which would require a measurable structure on the upward closed subsets of the measurable sets of the set of distributions over the state space.

The design of non-deterministic labelled Markov processes proceeds along this path: the set of measurable subsets of the measures on a measurable space is equipped with a σ-algebra (akin to defining a topology on the space of all closed subsets of a topological space), and measurability is defined through this σ-algebra [5], the *hit* σ*-algebra*. It turns out that one of the important issues here is composability of effectivity functions for the purpose of catering for the composition of games (or, more general, of actions). This is difficult to achieve with the concept of hit measurability, and it will be taken care of by the concept of t-measurability, which is introduced below.

Let us briefly discuss the issue of composability. Bind actions γ and δ to effectivity functions P_γ and P_δ; we want to model action $\gamma; \delta$, the sequential execution of γ and δ. Given a measurable set A, the sets $\beta(A, r) := \{\mu \mid \mu(A) > r\}$ collect