Theoretical Computer Science 625 (2016) 125–146

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

On natural deduction in classical first-order logic: Curry–Howard correspondence, strong normalization and Herbrand's theorem

Federico Aschieri [a,1], Margherita Zorzi [b,*,2]

[a] Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Austria
[b] Dipartimento di Informatica, Università di Verona, Italy

ARTICLE INFO

Article history:
Received 22 September 2014
Received in revised form 17 January 2016
Accepted 21 February 2016
Available online 26 February 2016
Communicated by M. Hofmann

Keywords:
Classical first-order logic
Natural deduction
Herbrand's theorem
Delimited exceptions
Curry–Howard correspondence

ABSTRACT

We present a new Curry–Howard correspondence for classical first-order natural deduction. We add to the lambda calculus an operator which represents, from the viewpoint of programming, a mechanism for raising and catching multiple exceptions, and from the viewpoint of logic, the excluded middle over arbitrary prenex formulas. The machinery will allow to extend the idea of learning – originally developed in Arithmetic – to pure logic. We prove that our typed calculus is strongly normalizing and show that proof terms for simply existential statements reduce to a list of individual terms forming an Herbrand disjunction. A by-product of our approach is a natural-deduction proof and a computational interpretation of Herbrand's Theorem.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

In the midst of an age of baffling paradoxes and contradictions, during the heat of a harsh controversy between opposed approaches to foundations of mathematics – infinitism vs. constructivism – a new and really penetrating insight was required to see a way out. Hilbert's proposed solution, at the beginning of twentieth century, was certainly deep and brilliant. According to him, there was no contradiction between classical and intuitionistic mathematics, because the ideal objects and principles that appear in classical reasoning can always be eliminated from proofs of concrete, incontestably meaningful statements. Hilbert's idea was made precise in his *epsilon elimination method* (see [25,22]), a systematic procedure to eliminate ideal objects from classical proofs and reduce every logical step to a concrete calculation. Hilbert's program was to show the termination of his method, or variants thereof, initially for first-order classical logic, then Peano Arithmetic and finally Analysis. As it turned out, Hilbert was right, and some termination proofs have been provided for example by Ackermann (for a modern proof see [24]) and Mints [23].

* Corresponding author.
E-mail address: margherita.zorzi@univr.it (M. Zorzi).

[1] This work was partially supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).
[2] Partially supported by LINTEL (Linear Techniques For The Analysis of Languages), https://sites.google.com/site/tolintel/.

http://dx.doi.org/10.1016/j.tcs.2016.02.028
0304-3975/© 2016 Elsevier B.V. All rights reserved.

1.1. Herbrand's and Kreisel's Theorems

After Hilbert, two other seminal results had been obtained stating that it is always possible to eliminate non-constructive reasoning in two important logical systems.

- The first one is *Herbrand's Theorem* [10], which says that if a simply existential statement $\exists\alpha\, \mathsf{P}$ is derivable in classical first-order logic from a set of purely universal premises, then there is a sequence of terms $m_1, m_2, \ldots, m_k$ such that the *Herbrand disjunction* $\mathsf{P}[m_1/\alpha] \vee \mathsf{P}[m_2/\alpha] \vee \ldots \vee \mathsf{P}[m_k/\alpha]$ is provable in classical propositional logic from a set of instances of the premises.
- The second one is *Kreisel's Theorem* [19], which says that if a simply existential formula $\exists\alpha\, \mathsf{P}$ is derivable in classical first-order Arithmetic, then it is derivable already in intuitionistic first-order Arithmetic. Using Kreisel's modified realizability [20] (or many other techniques), one can compute out of the intuitionistic proof a number n – a witness – such that $\mathsf{P}[n/\alpha]$ is true, whenever $\mathsf{P}[n/\alpha]$ it is closed.

Both Herbrand's and Kreisel's proof techniques are now obsolete, but the meaning of their results is as valid as ever, because it provides a theoretical justification for an important quest: the search for the constructive content of classical proofs. Herbrand's Theorem tells us what is the immediate computational content of classical first-order logic: the list of witnesses contained in any Herbrand disjunction. Kreisel's Theorem tells us what is the immediate computational content of first-order Arithmetic: the numeric witness for any provable existential statement. What is of great interest, in the light of those results, is to automatically transform proofs into programs in order to compute from any proof of any existential statement a suitable list of witnesses, in first-order logic, a single witness, in Arithmetic. In this paper, we shall address the first-order version of the problem – and propose a new solution.

1.2. Natural deduction and sequent calculus

The two most successful and most studied deductive systems for first-order logic are Gentzen's natural deduction [27] and Gentzen's sequent calculus [15,14]. The first elegant constructive proof of Herbrand's Theorem was indeed obtained as a corollary of Gentzen's Cut elimination Theorem. Today, that proof is still the most cited and the most used. On the contrary, we even failed to find in the literature a complete proof of Herbrand's Theorem using classical natural deduction. This is no coincidence, but yet another instance of the legendary duality between the two formalisms: as a matter of fact, some results are much more easily discovered and proved in the sequent calculus, while others are far more easily obtained in natural deduction. Since the time of Gentzen, natural deduction worked seamlessly for intuitionistic logic, and led to the discovery of the Curry–Howard correspondence [28], while sequent calculus was much more technically convenient in classical logic. As pointed out by [30], Gentzen's motivation for the creation of sequent calculus was indeed that he was not able to prove a meaningful normalization theorem for classical natural deduction, whilst he *was* for the intuitionistic case. It indeed took a surprisingly long time to discover suitable reduction rules for classical natural deduction systems with all connectives [17] (see also [7,28] for a more detailed history).

The great advantage of using natural deduction instead of sequent calculus is no mystery: it is *natural*! In Gentzen's own words, the main aim of natural deduction was to "*reproduce as precisely as possible the real logical reasoning in mathematical proofs*" [30]. Indeed, when logically solving non-trivial problems, humans adopt forward reasoning, which is more adapted to *proof-construction*: one starts from some observations, draws some consequences and gradually combines them so as to reach the goal. All of that can be elegantly represented in natural deduction. On the other hand, sequent calculus is more suitable for machine-like *proof-search*: one starts from the final goal and applies mechanically logical rules to reach axioms. As a consequence, when analyzing real mathematical proofs so to investigate their constructive content, one prefers to use natural deduction. Moreover, the reduction of a proof into normal form is nothing but the evaluation of a functional program, and so very easy to understand. The cut-elimination process, instead, is far more involved and difficult to follow. For example, the proof of Herbrand's Theorem by cut-elimination is deceptively simple: while it is rather obvious that the final cut-free proof contains an Herbrand disjunction, it is very painful to gain a step-by-step and clear understanding of how the corresponding list of witnesses has been produced.

1.3. Classical natural deduction: an exception-based Curry–Howard correspondence

We would like to endow classical first-order natural deduction with a *natural* set of reduction rules that also allows a *natural, seamless* proof of Herbrand's Theorem. As a corollary, this system would also have a simple and meaningful computational interpretation. Indeed, we believe that one can say to really understand a theorem when one is able to construct a proof of it that, a posteriori, appears completely natural, almost obvious. Usually, that happens when one has created a framework of concepts and methods that *explain* the theorem.

1.3.1. EM_1 and exceptions in arithmetic

If one wants to understand how is it possible that a classical proof has any computational content in the first place, the concept of *learning* is essential. It was a discovery by Hilbert that from classical proofs one can extract approximation

Download English Version:

https://daneshyari.com/en/article/433680

Download Persian Version:

https://daneshyari.com/article/433680

Daneshyari.com