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Lower bounds on the size of semi-quantum finite automata ☆

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ABSTRACT

In the literature, there exist several interesting hybrid models of finite automata which have both quantum and classical states. We call them semi-quantum finite automata. In this paper, we compare the descriptional power of these models and DFA. Specifically, we present a uniform method that gives a lower bound on the size of the three existing main models of semi-quantum finite automata, and this bound shows that semi-quantum finite automata can be at most exponentially more concise than DFA. Compared with a recent work [4], our method has the following two advantages: (i) it is much more concise; and (ii) it is universal, since it is applicable to the three existing main models of semi-quantum finite automata, instead of only one specific model.

1. Introduction

Quantum finite automata (QFA), as theoretical models for quantum computers with finite memory, have been explored by many researchers. So far, a variety of models of QFA have been introduced and explored to various degrees (one can refer to a review article [14] and the references therein). Among these QFA, there is a class of QFA that differ from others by consisting of two interactive components: a quantum component and a classical one. We call them *semi-quantum finite automata* in this paper. Examples of semi-quantum finite automata are *one-way QFA with control language* (CL-1QFA) [2], *one-way QFA together with classical states* (1QFAC) [13], and *one-way finite automata with quantum and classical states* (1QCFA) [15]. Here “one-way” means that the automaton’s tape head is required to move right on scanning each tape cell.

These semi-quantum finite automata have been proved to not only recognize all regular languages, but also show superiority over DFA with respect to descriptional power. For example, 1QCFA, CL-1QFA and 1QFAC were all shown to be much smaller than DFA in accepting some languages (solving some promise problems) [5,10,13,16]. In addition, a lower bound on the size of 1QFAC was given in [13], which stated that 1QFAC can be at most exponentially more concise than DFA, and the bound was shown to be tight by giving some languages witnessing this exponential gap. Size lower bounds were also reported for CL-1QFA in [4] and for 1QCFA in [3] (no detailed proof was given in [3] for the bound of 1QCFA), but they were not proved to be tight.

Specially, one can see that complex technical treatments were used in [4] to derive the bound for CL-1QFA and one may find that some key steps in [4] were confused such that the proof there may have some flaws, which will be explained more clearly in Section 4. It is also worth mentioning that the method used in [4] is tailored for CL-1QFA and is not easy to adopt to other models.

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Therefore, it is natural to ask: is there a uniform and simple method giving lower bounds on the size of the above three semi-quantum finite automata? This is possible, as 1QCFA, CL-1QFA and 1QFAC have the similar structure as shown in [8], where they were described in a uniform way: a semi-quantum finite automaton can be seen as a two-component communication systems comprising a quantum component and a classical one, and they differ from each other mainly in the specific communication pattern: classical-quantum, or quantum-classical, or two-way. It was also proved in [8] that the three models can be simulated by the model of QFA with mixed states and trace-preserving quantum operations (referred as MO-1gQFA) [9].

In this paper, based on the results from [8] we present a uniform method that gives a lower bound on the size of 1QCFA, CL-1QFA and 1QFAC, and this lower bound shows that they can be at most exponentially more concise than DFA. Specifically, we first obtain a lower bound on the size of MO-1gQFA and then apply it to the three hybrid models by using the relationship between them and MO-1gQFA. Compared with a recent work [4], our method is much more concise and universal, and it can be applied to the three existing main models of semi-quantum finite automata. In addition, our method may fix a potential mistake in [4] that will be pointed out in Section 4.

2. Preliminaries

Throughout this paper, A^* and $A^\dagger$ denote the conjugate and conjugate-transpose of a matrix (operator) A, respectively, and $\mathrm{Tr}(A)$ and $rank(A)$ denote the trace and rank of A, respectively. According to quantum mechanics, a quantum system is associated with a Hilbert space which is called the state space of the system. In this paper, we only consider finite dimensional spaces. A (mixed) state of a quantum system is represented by a density operator on its state space. A density operator ρ on $\mathcal{H}$ is a positive semi-definite linear operator such that $\mathrm{Tr}(\rho)=1$. If $rank(\rho)=1$, that is, $\rho=|\psi\rangle\langle\psi|$ for some $|\psi\rangle\in\mathcal{H}$, then ρ is called a pure state. Let $L(\mathcal{H})$ and $D(\mathcal{H})$ be the sets of linear operators and density operators on $\mathcal{H}$, respectively.

A trace-preserving quantum operation $\mathcal{E}$ on state space $\mathcal{H}$ is a linear map from $L(\mathcal{H})$ to itself that has an *operator-sum representation* as

$$\mathcal{E}(\rho)=\sum_k E_k\rho E_k^\dagger, \tag{1}$$

with the completeness condition $\sum_k E_k^\dagger E_k=I$, where $\{E_k\}$ are called operation elements of $\mathcal{E}$.

A general measurement is described by a collection $\{M_m\}$ of measurement operators, where the index m refers to the potential measurement outcome, satisfying the condition $\sum_m M_m^\dagger M_m=I$. If this measurement is performed on a state ρ, then the classical outcome m is obtained with the probability $p(m)=\mathrm{Tr}(M_m^\dagger M_m\rho)$, and the post-measurement state is

$$\frac{M_m\rho M_m^\dagger}{\sqrt{p(m)}}. \tag{2}$$

For the case that ρ is a pure state $|\psi\rangle$, that is, $\rho=|\psi\rangle\langle\psi|$, we have $p(m)=\|M_m|\psi\rangle\|^2$, and the state $|\psi\rangle$ "collapses" into the state

$$\frac{M_m|\psi\rangle}{\sqrt{p(m)}}. \tag{3}$$

A special case of general measurements is the projective measurement $\{P_m\}$ where P_m's are orthogonal projectors.

$A\in L(\mathcal{H})$ has the singular value decomposition [6,12] as follows:

$$A=\sum_{i=1}^{r}s_i|u_i\rangle\langle v_i|, \tag{4}$$

where $r=rank(A)$, $s_1,s_2,\ldots,s_r>0$ are called singular values of A, and $\{|v_i\rangle\}_{i=1}^r,\{|u_i\rangle\}_{i=1}^r\subset\mathcal{H}$ are two orthonormal sets.

The trace norm of $A\in L(\mathcal{H})$ is defined as $||A||_{tr}=\mathrm{Tr}\sqrt{A^\dagger A}$. By the singular value decomposition in (4), the trace norm can be characterized by singular values as

$$||A||_{tr}=\sum_i s_i. \tag{5}$$

Note that if A is positive semi-definite, then $||A||_{tr}=\mathrm{Tr}(A)$.

For $A,B\in L(\mathcal{H})$, the trace distance between them is

$$D(A,B)=||A-B||_{tr}. \tag{6}$$

The trace distance between two probability distributions $\{p_x\}$ and $\{q_x\}$ is