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Algebraic–coalgebraic recursion theory of history-dependent dynamical system models

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ABSTRACT

We investigate the common recursive structure of history-dependent dynamic models in science and engineering. We give formal semantics in terms of a hybrid algebraic–coalgebraic scheme, namely course-of-value iteration. This theoretical approach yields categories of observationally equivalent model representations with precise semantic relationships. Along the initial–final axis of these categories, history dependence can appear both literally and transformed into instantaneous state. The framework can be connected to philosophical and epistemological discourse on one side, and to algorithmic considerations for computational modeling on the other.

1. Introduction

Models of system dynamics are a cornerstone of science and engineering. They relate the future of a system to its present and/or past. An obvious qualitative distinction is whether observation of the present (*state*) alone suffices to predict or control the future, or whether information about the past (*history*) is necessary. This question can be discussed on the philosophical or on the mathematical level, the latter potentially leading to theoretical frameworks and tools for the working scientist and engineer.

In this paper, we explore the mathematical option, and present a formalization that puts the two model classes on equal footing. Specifically, they shall be demonstrated to form not a dichotomy, but a continuum along the initial–final axis of suitable categories of models, constructed from first principles of algebraic–coalgebraic recursion theory.

In philosophical terms, this framework gives precise semantic relationships between more and less history-dependent models, and thus renders the two most common objections against history-dependent modeling obsolete: arguments from naïve reductionism (invoking Laplace's Daemon) and arguments from parsimony (invoking Occam's Razor).

We begin with motivating examples that demonstrate the pervasive occurrence, and the vastly different relative reputability, of the two modeling approaches across various disciplines. The purpose of this digression is to demonstrate the broad applicability of our proposed framework, which is hard to see directly from the austere formal results.

Readers more narrowly interested in theoretical matters are encouraged to skip ahead to Section 2. There we review the relevant concepts from (co)algebraic recursion theory, and illustrate their respective use and mutual relationship with simple examples from both algorithmics and scientific modeling. In Section 3, we outline a general categorial framework for realizations of history-dependent recursive functions in terms of state, which may or may not be conceived as instantaneous. In

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Section 4, we refine the theory to a more specific case that covers autonomous discrete-time systems in scientific modeling as well as enumerations of Fibonacci-like sequences. Finally we discuss some future potential of the approach.

1.1. Digression: philosophical reasoning

The problem of history in models has troubled philosophers of science, and withstood efforts to be understood properly in terms of non-formal reasoning. This is particularly manifest in environmental sciences: Even though natural history is part of science, its biological and ecological content is often studied and taught in terms of stories which are interpreted and evaluated only in hindsight as adaptations (survival of the fittest), rather than reduced to objective properties of state. Models of evolution of *artificial* life consequently have difficulties defining what is at the core of history. Pragmatics and demonstrations prevail over formalization and theory [1–3].

In the present paper, we conjecture that the different degree of success of mathematical formalization for systems with and without history dependence is itself for "historical" rather than logical reasons; present-day mathematics is able to express both in commensurable terms.

The following example is characteristic of the traditional difficulties in understanding, and a major source of inspiration for our work:

The biologist Rosen proposed a general account of scientific modeling in his last book [4]. It is founded on the postulate that the formal aspects of models, that is their algebra and logic, hinge on recursive regularities in data:

> *We say that the function f, defined in this manner, is* recursively defined*; its successive values are obtained, not by evaluating it at the successive numbers in its* domain*, but by applying a fixed operation or mapping T to its* preceding value. [...] *This apparently trivial situation is the germ on which the state concept, and hence, contemporary theoretical science itself, rests.* [4, emphasis in original]

The manner of recursive regularity described here is of course known as "primitive" recursion to the mathematician.

However, Rosen is overly critical with regard to forms of recursive regularity that do not fit the primitive pattern:

> *I might point out that there are more complicated situations that in a formal sense also involve recursion, entailment between chronicle entries. Perhaps the most familiar is embodied in the well-known Fibonacci sequences; in these, the first* two *members of the sequence are arbitrary, and all successive elements of the sequence are entailed from these via rules of the form* [$a_3 = T(a_1, a_2)$ etc.]. *However, such modes of entailment (which, heuristically speaking, involve "time lags") are not suitable for encoding a concept of state. See, e.g., AS* [5, i.e.]. [4]

An algorithm for the enumeration of the Fibonacci sequence that is based purely and soundly on instantaneous state is evident to the computer scientist. Thus the cited special case is easily refuted; confer Fig. 3. But a general verdict on "such modes" requires more sophisticated reasoning. The remainder of the present paper is essentially a careful explication of the relevant concepts.

It is no accident that the philosophical investigation of state-based models has been advanced by a biologist: The realm of biology poses several challenges to dynamical system models. On the one hand, organisms are individuals with unique evolutionary and developmental histories. On the other hand, most modeling approaches in biology seek to identify the "living state" along with a mechanistic explanation. Unlike non-living systems, many behavioral features of organisms have pragmatically been characterized as purposeful, giving rise to several philosophical and modeling approaches how to accommodate (apparently) teleological features in natural history [6, e.g.].

Rosen has pioneered the use of category theory for defining and modeling life. However, categorial notions remain metaphorical in many of his arguments. The present paper is part of a series [7,8] that reports on our efforts to demonstrate their viability in full formal detail.

1.2. Basic scientific example: simple harmonic oscillator

A simple harmonic oscillator is an ideal point mass m moving frictionlessly along a line and acted on by a restoring force proportional, with positive coefficient k, to its displacement x. Consider an empirical study of (real approximations of) many such systems that produces a collection of time series, consisting of displacements observed equidistantly in time with some small delay δ. The search for time-invariant patterns, untainted by theoretical presuppositions, in these data might reveal that with good accuracy, the ratio between each displacement and the arithmetic mean of its neighbors is determined by the given parameters, according to the formula:

$$\frac{x_{t-\delta} + x_{t+\delta}}{2x_t} = 1 - \frac{k}{2m}\delta^2 \qquad \text{for all } t \tag{1}$$

In the form of a dynamic model, where the future is displayed as a function of the present (and past), we obtain for any *three* displacements the following model formula:

$$x_{t+\delta} = \left(2 - \tfrac{k}{m}\delta^2\right)x_t - x_{t-\delta} \tag{2a}$$