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Lower and upper bounds for long induced paths in 3-connected planar graphs ☆

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ABSTRACT

Let G be a 3-connected planar graph with n vertices and let $p(G)$ be the maximum number of vertices of an induced subgraph of G that is a path. Substantially improving previous results, we prove that $p(G) \geq \frac{\log n}{12 \log \log n}$. To demonstrate the tightness of this bound, we notice that the above inequality implies $p(G) \in \Omega((\log_2 n)^{1-\varepsilon})$, where ε is any positive constant smaller than 1, and describe an infinite family of planar graphs for which $p(G) \in O(\log n)$. As a byproduct of our research, we prove a result of independent interest: Every 3-connected planar graph with n vertices contains an induced subgraph that is outerplanar and connected and that contains at least $\sqrt[3]{n}$ vertices. The proofs in the paper are constructive and give rise to $O(n)$-time algorithms.

1. Introduction

Determining whether a graph has a large subset of vertices that induce a path is a well studied problem motivated, for example, by applications in the performance analysis of large communication and neural networks (see, e.g., [8]). Garey and Johnson [7] prove the NP-completeness of computing the longest induced path for general graphs and Lund and Yannakakis [15] show that this maximization problem is not approximable within $O(n^{1-\varepsilon})$ for any positive constant ε. Exact algorithms that have polynomial time complexity are also known for specific graph families (see, e.g. [9,8,12,13]), while several papers study the complexity of various network optimization problems under the assumption that the length of the longest induced path is bounded (see, e.g. [2,14,18]).

From a graph theoretical point of view, Erdös, Saks, and Sós [5] prove a lower bound on the length of the longest induced path in a connected graph G in terms of the radius of G. They show that $p(G) \geq 2r(G) - 1$, where $p(G)$ denotes the maximum number of vertices in an induced path of a connected graph G and $r(G)$ is the radius of G. Since, in general, the radius is not bounded from below by a function of n, the result by Erdös, Saks, and Sós naturally raises the question about whether the length of the longest path tends to infinity as the size of the graph tends to infinity. Clearly, this question becomes interesting for graphs that are not too dense; namely, the longest induced path in a complete graph consists of a single edge.

For 3-connected planar graphs, a positive answer to the above question is a consequence of a Ramsey-type result by Böhme et al. [3]. Namely, in [3] it is proved that for every positive integers k, r, s, there exists an integer $n = n(k, r, s)$ such

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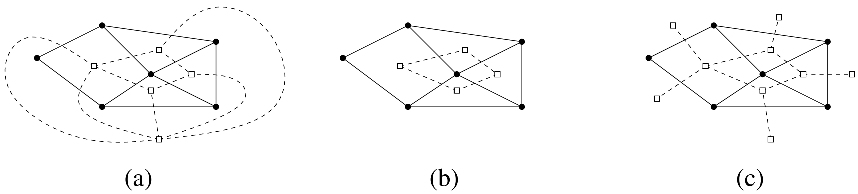

Fig. 1. (a) A graph G and its dual graph G^*: the black dots are the vertices of G, while the white squares are those of G^*; the solid edges are the edges of G, while the dashed edges are those of G^*. (b) The weak dual of G. (c) The extended dual of G.

that any k-connected graph with at least n vertices has either an induced path of length s or a subdivision of $K_{k,r}$. Hence, by setting $k = r = 3$, we obtain that for every positive integer s there exists a sufficiently large 3-connected planar graph having an induced path of length s. However, the construction by Böhme et al. does not yield an explicit function that defines a lower bound on the length of the longest induced path, which can be found in a paper by Arocha and Valencia [1]. Let G be a 3-connected planar graph with n vertices and having maximum degree Δ; Arocha and Valencia observe that if Δ is bounded by a constant, then G has a diameter (and hence an induced path) consisting of at least $\log_\Delta n$ vertices. If otherwise Δ is not a constant with n, then G has an induced outerplanar graph with at least Δ vertices from which an induced path with at least $\sqrt{\log_3 \Delta}$ vertices can be extracted.

The main contribution of this paper is to find bounds for $p(G)$ that improve those in [1,3]. Similar to Arocha and Valencia, we first construct an induced outerplanar graph H and then compute an induced path in H; however, our approach finds significantly larger outerplanar graphs and significantly larger paths than those described in [1]. More precisely, the results in this paper can be listed as follows.

- We show that a 3-connected planar graph G with n vertices has an induced subgraph with at least $\sqrt[3]{n}$ vertices that is outerplanar and connected. In the case that the external face of G is a 3-cycle, the induced outerplanar graph is 2-connected. This result is of independent interest because it improves a previous bound by Goaoc et al. [11] who show that a maximal planar graph with n vertices has a 2-connected induced outerplanar subgraph with $\Omega(\frac{\log n}{\log\log n})$ vertices.
- We prove that every 3-connected planar graph G with n vertices has an induced path with at least $\frac{\log n}{12 \log\log n}$ vertices. We also prove that for every n there exists a 3-connected planar graph G with n vertices such that the longest induced path of G has at most $2\log_3(2n-5)+3$ vertices.

We also show that, in asymptotic terms, the gap between the upper and lower bounds described in the last item is arbitrarily small. Namely, a consequence of the above result is that for any given positive constant ε smaller than 1 and for n that tends to infinity, every 3-connected planar graph has an induced path with at least $(2\log_3(2n-5)+3)^{1-\varepsilon}$ vertices.

Our arguments combine various techniques. We use the extension of Schnyder woods to 3-connected planar graphs (see, e.g. [4,6]) to prove that there exist three partial orders by which any two vertices of 3-connected planar graph can be compared. We then exploit Mirsky's theorem [16] to extract a large induced outerplanar graph H; finally, we analyze the structure of the extended dual of H to compute a long induced path. The proofs for the lower bound are constructive and give rise to a linear-time algorithm to compute a long induced path in a 3-connected planar graph.

The rest of the paper is organized as follows. Preliminaries are in Section 2. The computation of a large induced outerplanar graph is described in Section 3. Lower and upper bounds on the length of the longest induced path are given in Sections 4 and 5. Finally, conclusions and open problems can be found in Section 6.

2. Preliminaries

A subgraph H of a graph G is said to be *induced* if, for any pair of vertices u and v of H, (u, v) is an edge of H if and only if (u, v) is an edge of G. We denote by $p(G)$ the number of vertices of the longest induced path of G.

The *connectivity* of a connected graph is the minimum number of vertices whose removal results in a disconnected graph or a single vertex graph. A graph is *k-connected* if its connectivity is at least k ($k \geq 1$). Notice that a k-connected graph has at least $k+1$ vertices. A *cutvertex* is a vertex whose removal disconnects the graph. Let G be a 1-connected graph (also called a connected graph); the maximal subgraphs of G not containing a cutvertex is a 2-*connected component* of G. Notice that a 2-connected component of a connected graph is either a 2-connected subgraph or a single edge.

A graph G is planar if it can be drawn in the plane without edge crossings. A planar drawing Γ partitions the plane into topologically connected regions called *faces*; the unbounded region is the *external* face. A planar drawing of a planar graph determines a circular ordering of the edges around each vertex. The cyclic ordering of the edges around each vertex of Γ together with a choice of the external face is a *planar embedding* of G. A *plane graph* is a graph with a fixed planar embedding. The boundary of the external face of a plane graph G will be also called the *external boundary* of G.

The *dual graph* G^d of a given plane graph G is a plane multigraph that has a vertex corresponding to each face of G, and an edge joining two vertices corresponding to neighboring faces of G (refer to Fig. 1(a)). The *weak dual* G^w of G is the graph obtained from G^d after removing the vertex that corresponds to the external face of G (see Fig. 1(b)). The *extended*