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Finding good 2-partitions of digraphs I. Hereditary properties

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ABSTRACT

We study the complexity of deciding whether a given digraph D has a vertex-partition into two disjoint subdigraphs with given structural properties. Let $\mathcal{H}$ and $\mathcal{E}$ denote the following two sets of natural properties of digraphs: $\mathcal{H}$ = {acyclic, complete, arcless, oriented (no 2-cycle), semicomplete, symmetric, tournament} and $\mathcal{E}$ = {strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching}. In this paper, we determine the complexity of deciding, for any fixed pair of positive integers k_1, k_2, whether a given digraph has a vertex partition into two digraphs D_1, D_2 such that $|V(D_i)| \geq k_i$ and D_i has property $\mathbb{P}_i$ for $i = 1, 2$ when $\mathbb{P}_1 \in \mathcal{H}$ and $\mathbb{P}_2 \in \mathcal{H} \cup \mathcal{E}$. We also classify the complexity of the same problems when restricted to strongly connected digraphs.
The complexity of the 2-partition problems where both $\mathbb{P}_1$ and $\mathbb{P}_2$ are in $\mathcal{E}$ is determined in the companion paper [2].

1. Introduction

A k-**partition** of a (di)graph D is a partition of $V(D)$ into k disjoint sets. Let $\mathbb{P}_1, \mathbb{P}_2$ be two (di)graph properties, then a $(\mathbb{P}_1, \mathbb{P}_2)$-**partition** of a (di)graph D is a 2-partition (V_1, V_2) where V_1 induces a (di)graph with property $\mathbb{P}_1$ and V_2 a (di)graph with property $\mathbb{P}_2$. For example a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least 1.

There are many papers dealing with vertex-partition problems on (di)graphs. Examples include [1,4,5,7–16,18,20–23]. Important examples for undirected graphs are bipartite graphs, that is, those graphs having a 2-partition into two independent sets and split graphs, that is, those graphs having a 2-partition into a clique and an independent set [8]. It is well known and easy to show that there are linear-time algorithms for deciding whether a graph is bipartite, respectively, a split graph. The **dichromatic number** of a digraph D [17] is the minimum number k such that D has a k-partition where each

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set in the partition induces an acyclic digraph. This is a natural analogue of the chromatic number for undirected graphs as a graph G has chromatic number k if and only if the symmetric digraph $\overleftrightarrow{G}$, that we obtain from G by replacing every edge by a directed 2-cycle, has dichromatic number k. Contrary to the case of undirected graphs, it is already NP-complete to decide whether a digraph has dichromatic number 2 [5] (see also the proof of Theorem 4.4).

A set of vertices X in a digraph D is a **feedback vertex set** if $D - X$ is acyclic. If we wish to study feedback vertex sets with a certain property $\mathbb{P}$, this is the same as studying the ($\mathbb{P}$, acyclic)-partition problem. For example we may seek a feedback vertex set that induces an acyclic digraph and that is the (acyclic, acyclic)-partition problem which is the same as asking whether D has dichromatic number at most 2 and hence is NP-complete as noted above. On the other hand, if we want the feedback vertex set to be connected, we obtain the (connected, acyclic)-partition problem which is polynomial-time solvable as we show in Corollary 3.2.

In this paper and its companion paper [2] we give a complete characterization for the complexity of $(\mathbb{P}_1, \mathbb{P}_2)$-partition problems when $\mathbb{P}_1, \mathbb{P}_2$ are one of the following properties: acyclic, complete, independent (no arcs), oriented (no directed 2-cycle), semicomplete, tournament, symmetric (if two vertices are adjacent, then they induce a directed 2-cycle), strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching. All of these 15 properties are natural properties of digraphs (as we already indicated above, symmetric digraphs correspond to undirected graphs). For each of them, it can be checked in linear time whether the given digraph has this property. Hence all the 120 distinct 2-partition problems are in NP.

Several of these 120 $(\mathbb{P}_1, \mathbb{P}_2)$-partition problems are NP-complete and some results are surprising. For example, in [2], we show that the $(\delta^+ \geq 1, \delta \geq 1)$-partition problem is NP-complete. Some other problems are polynomial-time solvable because under certain conditions there are trivial $(\mathbb{P}_1, \mathbb{P}_2)$-partitions (V_1, V_2) with $|V_1| = 1$ (or $|V_2| = 1$). Therefore, in order to avoid such trivial partitions we consider $[k_1, k_2]$**-partitions**, that is, partitions (V_1, V_2) of V such that $|V_1| \geq k_1$ and $|V_2| \geq k_2$. Consequently, for each pair of above-mentioned properties and all pairs (k_1, k_2) of positive integers, we consider the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem, which consists in deciding whether a given digraph D has a $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition. When $k_1 = k_2 = 1$ we usually just write $(\mathbb{P}_1, \mathbb{P}_2)$-partition.

It might seem to be a lot of work but we are able to structure the approach in such a way that we can handle all the cases, especially most of the polynomial-time solvable ones, effectively. The results, including those from [2], are summarized in Table 1.

The paper is organized as follows. We first introduce the necessary terminology, and show that the properties in the classes $\mathcal{H}$ and $\mathcal{E}$, which we introduced in the abstract, are checkable and either hereditary or enumerable properties. These are defined below. Then in Section 3, we show that if $\mathbb{P}_1$ is hereditary and $\mathbb{P}_2$ is enumerable, then for any k_1, k_2, the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem is polynomial-time solvable. In Section 4, we determine the complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem for all possible pairs $(\mathbb{P}_1, \mathbb{P}_2)$ of elements in $\mathcal{H}$. The complexity of the problem for all possible pairs $(\mathbb{P}_1, \mathbb{P}_2)$ of elements in $\mathcal{E}$ is determined in the companion paper [2]. The results are summarized in Table 1. The grey cells correspond to results proved in [2].

Table 1
Complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem for some properties $\mathbb{P}_1, \mathbb{P}_2$.

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	$\mathbb{B}^+$	$\mathbb{B}^-$	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	$\mathbb{A}$	$\mathbb{C}$	$\mathbb{X}$
strong	NPc	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc	P	P	P
conn.	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\mathbb{B}^+$	NPc^R	P	P	NPc	P	NPc	P	NPc	P	P	P
$\mathbb{B}^-$	NPc^R	P	NPc	P	P	P	NPc	NPc	P	P	P
$\delta \geq 1$	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\delta^+ \geq 1$	NPc^R	NPc	NPc	P	NPc	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc^R	NPc	P	NPc	NPc	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	NPc	NPc	NPc	NPc	NPc	NPc	NPc	P	P	P
$\mathbb{A}$	P	P	P	P	P	P	P	P	NPc	P	NPc
$\mathbb{C}$	P	P	P	P	P	P	P	P	P	P	P
$\mathbb{X}$	P	P	P	P	P	P	P	P	NPc	P	P

Properties: conn.: connected; $\mathbb{B}^+$: out-branchable; $\mathbb{B}^-$: in-branchable; $\mathbb{A}$: acyclic; $\mathbb{C}$: complete; $\mathbb{X}$: any property in 'being independent', 'being oriented', 'being semi-complete', 'being a tournament' and 'being symmetric'.

Complexities: P: polynomial-time solvable; NPc: NP-complete for all values of k_1, k_2; NPc^L: NP-complete for $k_1 \geq 2$, and polynomial-time solvable for $k_1 = 1$. NPc^R: NP-complete for $k_2 \geq 2$, and polynomial-time solvable for $k_2 = 1$.

All the NP-completeness proofs given in this paper are also valid if we restrict the input digraph to be strongly connected. However, for some partition problems with two enumerable properties, the complexity is sometimes different when we restrict to strongly connected digraphs as shown in [2]. The complexity results of the problems restricted to strongly connected digraphs are summarized in Table 2. The grey cells correspond to results proved in [2].